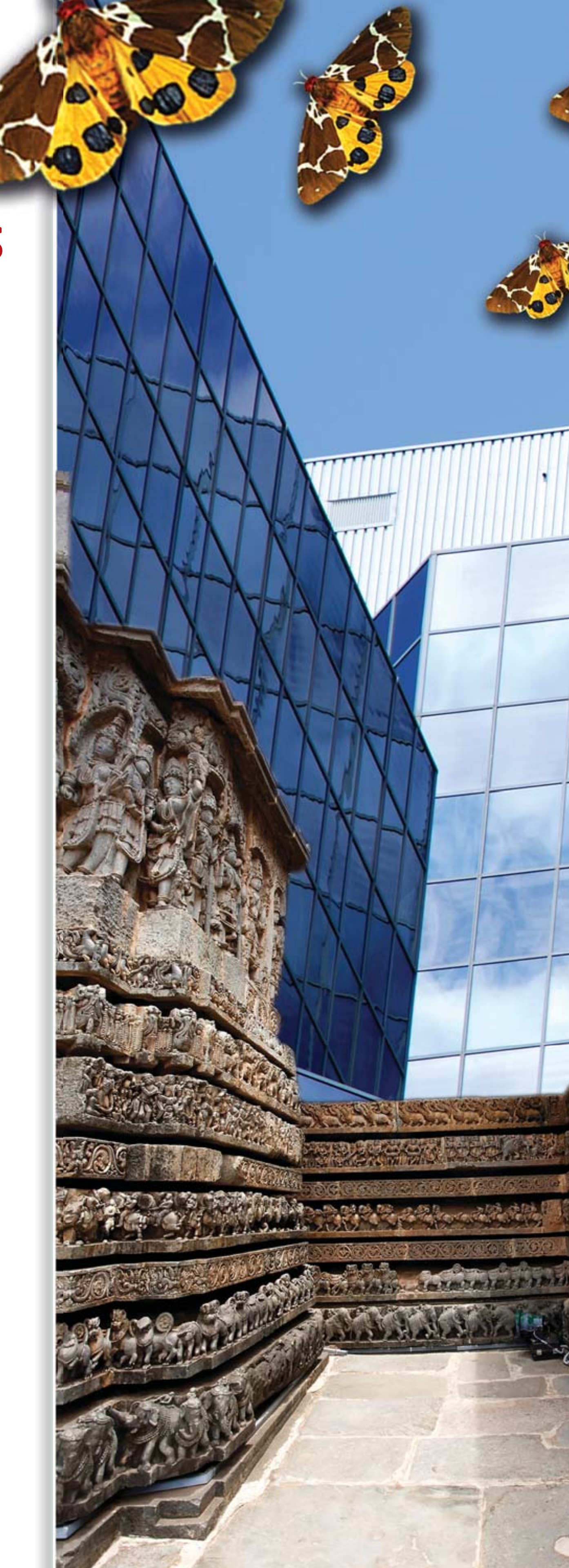
CHAPTER 1

Function Transformations



Mathematical shapes are found in architecture, bridges, containers, jewellery, games, decorations, art, and nature. Designs that are repeated, reflected, stretched, or transformed in some way are pleasing to the eye and capture our imagination.

In this chapter, you will explore the mathematical relationship between a function and its transformed graph. Throughout the chapter, you will explore how functions are transformed and develop strategies for relating complex functions to simpler functions.

Did You Know?

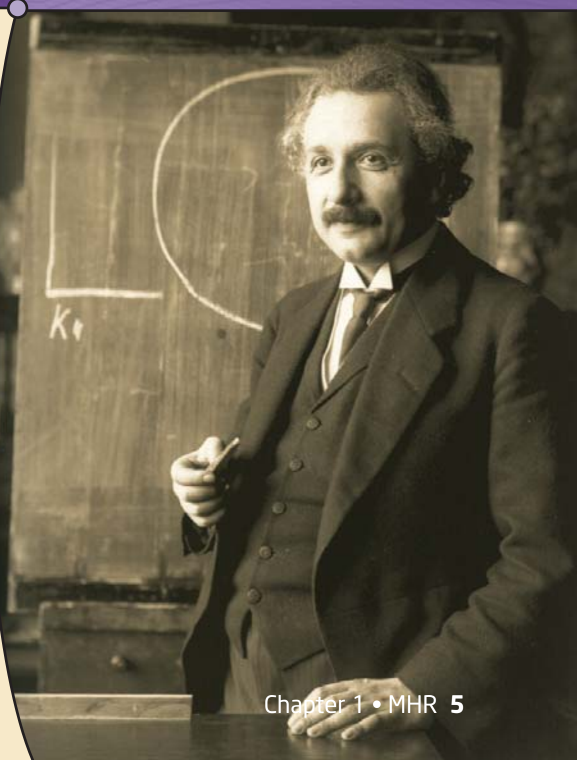
Albert Einstein (1879–1955) is often regarded as the father of modern physics. He won the Nobel Prize for Physics in 1921 for "his services to Theoretical Physics, and especially for his discovery of the law of the photoelectric effect." The Lorentz transformations are an important part of Einstein's theory of relativity.

Key Terms

- transformation
- mapping
- translation
- image point
- reflection
- invariant point
- stretch
- inverse of a function
- horizontal line test

Career Link

A physicist is a scientist who studies the natural world, from sub-atomic particles to matters of the universe. Some physicists focus on theoretical areas, while others apply their knowledge of physics to practical areas, such as the development of advanced materials and electronic and optical devices. Some physicists observe, measure, interpret, and develop theories to explain celestial and physical phenomena using mathematics. Physicists use mathematical functions to make numerical and algebraic computations easier.



Web Link

To find out more about the career of a physicist, go to www.mcgrawhill.ca/school/learningcentres and follow the links.

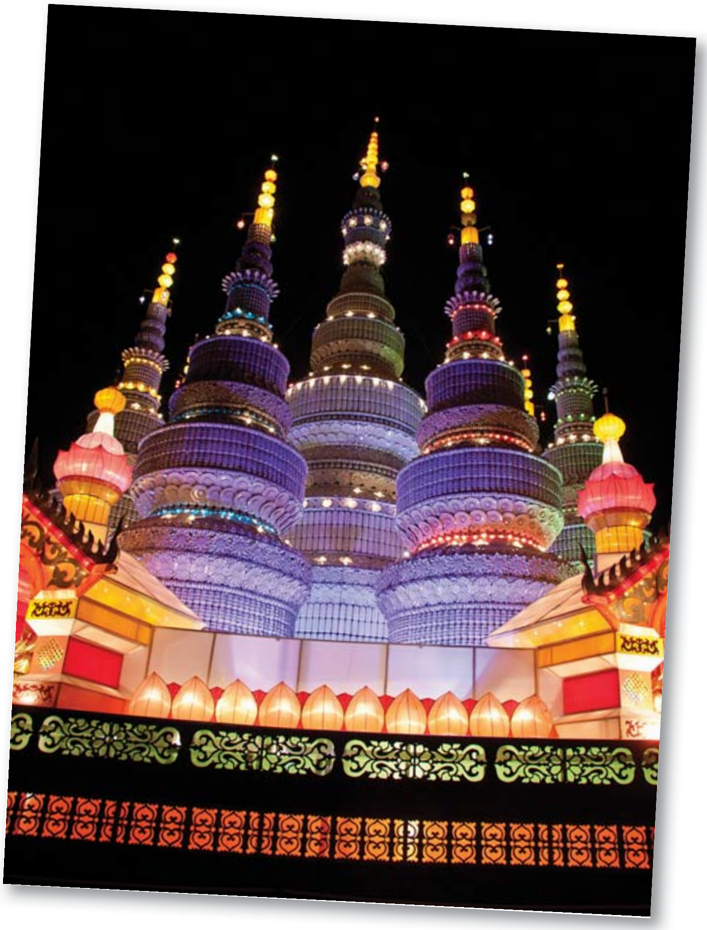
1.1

Horizontal and Vertical Translations

Focus on...

- determining the effects of h and k in $y - k = f(x - h)$ on the graph of $y = f(x)$
- sketching the graph of $y - k = f(x - h)$ for given values of h and k, given the graph of $y = f(x)$
- writing the equation of a function whose graph is a vertical and/or horizontal translation of the graph of $y = f(x)$

A linear frieze pattern is a decorative pattern in which a section of the pattern repeats along a straight line. These patterns often occur in border decorations and textiles. Frieze patterns are also used by artists, craftspeople, musicians, choreographers, and mathematicians. Can you think of places where you have seen a frieze pattern?



Lantern Festival in China

Investigate Vertical and Horizontal Translations

Materials

- grid paper

A: Compare the Graphs of $y = f(x)$ and $y - k = f(x)$

1. Consider the function $f(x) = |x|$.

a) Use a table of values to compare the output values for $y = f(x)$, $y = f(x) + 3$, and $y = f(x) - 3$ given input values of -3, -2, -1, 0, 1, 2, and 3.

b) Graph the functions on the same set of coordinate axes.

2. a) Describe how the graphs of $y = f(x) + 3$ and $y = f(x) - 3$ compare to the graph of $y = f(x)$.

b) Relative to the graph of $y = f(x)$, what information about the graph of $y = f(x) + k$ does k provide?

3. Would the relationship between the graphs of $y = f(x)$ and $y = f(x) + k$ change if $f(x) = x$ or $f(x) = x^2$? Explain.

B: Compare the Graphs of $y = f(x)$ and $y = f(x - h)$

4. Consider the function $f(x) = |x|$.

a) Use a table of values to compare the output values for $y = f(x)$, $y = f(x + 3)$, and $y = f(x - 3)$ given input values of -9, -6, -3, 0, 3, 6, and 9.

b) Graph the functions on the same set of coordinate axes.

5. a) Describe how the graphs of $y = f(x + 3)$ and $y = f(x - 3)$ compare to the graph of $y = f(x)$.

b) Relative to the graph of $y = f(x)$, what information about the graph of $y = f(x - h)$ does h provide?

6. Would the relationship between the graphs of $y = f(x)$ and $y = f(x - h)$ change if $f(x) = x$ or $f(x) = x^2$? Explain.

Reflect and Respond

7. How is the graph of a function $y = f(x)$ related to the graph of $y = f(x) + k$ when $k > 0$? when $k < 0$?

8. How is the graph of a function $y = f(x)$ related to the graph of $y = f(x - h)$ when $h > 0$? when $h < 0$?

9. Describe how the parameters h and k affect the properties of the graph of a function. Consider such things as shape, orientation, x-intercepts and y-intercept, domain, and range.

Link the Ideas

A **transformation** of a function alters the equation and any combination of the location, shape, and orientation of the graph.

Points on the original graph correspond to points on the transformed, or image, graph. The relationship between these sets of points can be called a **mapping**.

Mapping notation can be used to show a relationship between the coordinates of a set of points, (x, y), and the coordinates of a corresponding set of points, $(x, y + 3)$, for example, as $(x, y) \rightarrow (x, y + 3)$.

Did You Know?

Mapping notation is an alternate notation for function notation. For example, $f(x) = 3x + 4$ can be written as $f: x \rightarrow 3x + 4$. This is read as "f is a function that maps x to $3x + 4$."

transformation

- a change made to a figure or a relation such that the figure or the graph of the relation is shifted or changed in shape

mapping

- the relating of one set of points to another set of points so that each point in the original set corresponds to exactly one point in the image set

translation

- a slide transformation that results in a shift of a graph without changing its shape or orientation
- vertical and horizontal translations are types of transformations with equations of the forms $y - k = f(x)$ and $y = f(x - h)$, respectively
- a translated graph is congruent to the original graph

One type of transformation is a **translation**. A translation can move the graph of a function up, down, left, or right. A translation occurs when the location of a graph changes but not its shape or orientation.

Example 1

Graph Translations of the Form $y - k = f(x)$ and $y = f(x - h)$

a) Graph the functions $y = x^2$, $y - 2 = x^2$, and $y = (x - 5)^2$ on the same set of coordinate axes.

b) Describe how the graphs of $y - 2 = x^2$ and $y = (x - 5)^2$ compare to the graph of $y = x^2$.

Solution

a) The notation $y - k = f(x)$ is often used instead of $y = f(x) + k$ to emphasize that this is a transformation on y. In this case, the base function is $f(x) = x^2$ and the value of k is 2.

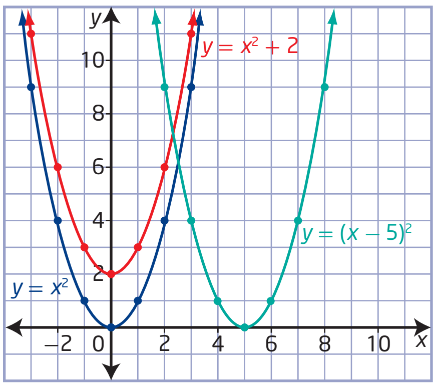
The notation $y = f(x - h)$ shows that this is a transformation on x. In this case, the base function is $f(x) = x^2$ and the value of h is 5.

Rearrange equations as needed and use tables of values to help you graph the functions.

x	$y = x^2$
−3	9
−2	4
−1	1
0	0
1	1
2	4
3	9

x	$y = x^2 + 2$
−3	11
−2	6
−1	3
0	2
1	3
2	6
3	11

x	$y = (x - 5)^2$
2	9
3	4
4	1
5	0
6	1
7	4
8	9



For $y = x^2 + 2$, the input values are the same but the output values change. Each point (x, y) on the graph of $y = x^2$ is transformed to $(x, y + 2)$.

For $y = (x - 5)^2$, to maintain the same output values as the base function table, the input values are different. Every point (x, y) on the graph of $y = x^2$ is transformed to $(x + 5, y)$. How do the input changes relate to the translation direction?

b) The transformed graphs are congruent to the graph of $y = x^2$.

Each point (x, y) on the graph of $y = x^2$ is transformed to become the point $(x, y + 2)$ on the graph of $y - 2 = x^2$. Using mapping notation, $(x, y) \rightarrow (x, y + 2)$.

Therefore, the graph of $y - 2 = x^2$ is the graph of $y = x^2$ translated vertically 2 units up.

Each point (x, y) on the graph of $y = x^2$ is transformed to become the point $(x + 5, y)$ on the graph of $y = (x - 5)^2$. In mapping notation, $(x, y) \rightarrow (x + 5, y)$.

Therefore, the graph of $y = (x - 5)^2$ is the graph of $y = x^2$ translated horizontally 5 units to the right.

Your Turn

How do the graphs of $y + 1 = x^2$ and $y = (x + 3)^2$ compare to the graph of $y = x^2$? Justify your reasoning.

Example 2

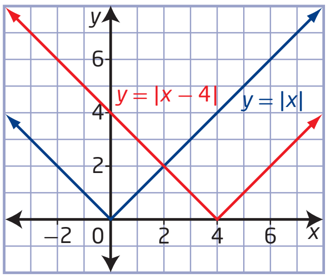
Horizontal and Vertical Translations

Sketch the graph of $y = |x - 4| + 3$.

Solution

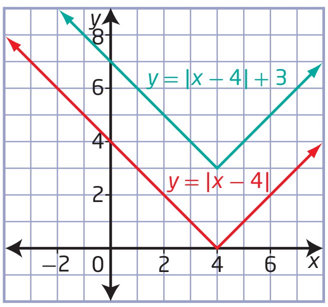
For $y = |x - 4| + 3$, $h = 4$ and $k = -3$.

- Start with a sketch of the graph of the base function $y = |x|$, using key points.
- Apply the horizontal translation of 4 units to the right to obtain the graph of $y = |x - 4|$.



To ensure an accurate sketch of a transformed function, translate key points on the base function first.

- Apply the vertical translation of 3 units up to $y = |x - 4|$ to obtain the graph of $y = |x - 4| + 3$.



Would the graph be in the correct location if the order of the translations were reversed?

The point (0, 0) on the function $y = |x|$ is transformed to become the point (4, 3). In general, the transformation can be described as $(x, y) \rightarrow (x + 4, y + 3)$.

Your Turn

Sketch the graph of $y = (x + 5)^2 - 2$.

Did You Know?

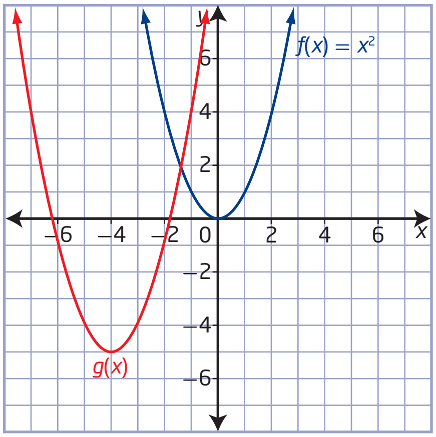
Key points are points on a graph that give important information, such as the x-intercepts, the y-intercept, the maximum, and the minimum.

Example 3

Determine the Equation of a Translated Function

Describe the translation that has been applied to the graph of $f(x)$ to obtain the graph of $g(x)$. Determine the equation of the translated function in the form $y - k = f(x - h)$.

a)



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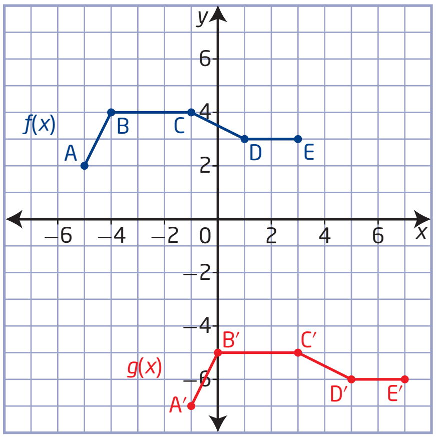


It is a common convention to use a prime (′) next to each letter representing an image point.

Solution

a) The base function is $f(x) = x^2$. Choose key points on the graph of $f(x) = x^2$ and locate the corresponding **image points** on the graph of $g(x)$.

image point

- the point that is the result of a transformation of a point on the original graph

$f(x)$		$g(x)$
(0, 0)	→	(−4, −5)
(−1, 1)	→	(−5, −4)
(1, 1)	→	(−3, −4)
(−2, 4)	→	(−6, −1)
(2, 4)	→	(−2, −1)
(x, y)	→	$(x - 4, y - 5)$

For a horizontal translation and a vertical translation where every point (x, y) on the graph of $y = f(x)$ is transformed to $(x + h, y + k)$, the equation of the transformed graph is of the form $y - k = f(x - h)$.

To obtain the graph of $g(x)$, the graph of $f(x) = x^2$ has been translated 4 units to the left and 5 units down. So, $h = -4$ and $k = -5$.

To write the equation in the form $y - k = f(x - h)$, substitute -4 for h and -5 for k.

$y + 5 = f(x + 4)$

b) Begin with key points on the graph of $f(x)$. Locate the corresponding image points.

$f(x)$		$g(x)$
A(−5, 2)	→	A′(−1, −7)
B(−4, 4)	→	B′(0, −5)
C(−1, 4)	→	C′(3, −5)
D(1, 3)	→	D′(5, −6)
E(3, 3)	→	E′(7, −6)
(x, y)	→	$(x + 4, y - 9)$

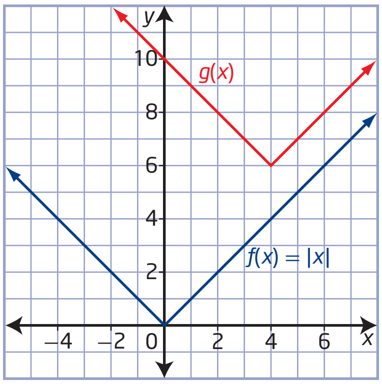
To obtain the graph of $g(x)$, the graph of $f(x)$ has been translated 4 units to the right and 9 units down. Substitute $h = 4$ and $k = -9$ into the equation of the form $y - k = f(x - h)$:

$y + 9 = f(x - 4)$

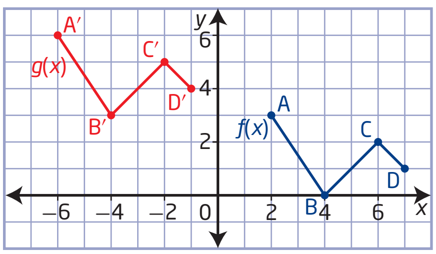
Your Turn

Describe the translation that has been applied to the graph of $f(x)$ to obtain the graph of $g(x)$. Determine the equation of the translated function in the form $y - k = f(x - h)$.

a)



b)

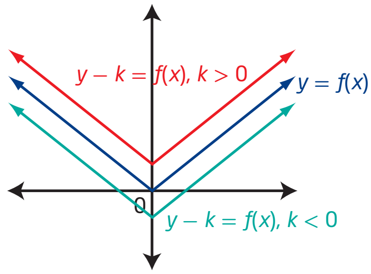
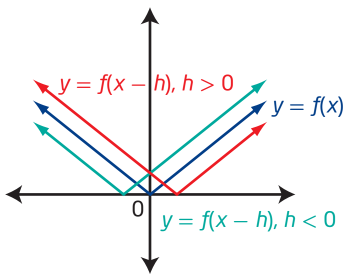


Did You Know?

In Pre-Calculus 11, you graphed quadratic functions of the form $y = (x - p)^2 + q$ by considering transformations from the graph of $y = x^2$. In $y = (x - p)^2 + q$, the parameter p determines the horizontal translation and the parameter q determines the vertical translation of the graph. In this unit, the parameters for horizontal and vertical translations are represented by h and k, respectively.

Key Ideas

- Translations are transformations that shift all points on the graph of a function up, down, left, and right without changing the shape or orientation of the graph.
- The table summarizes translations of the function $y = f(x)$.

Function	Transformation from $y = f(x)$	Mapping	Example
$y - k = f(x)$ or $y = f(x) + k$	A vertical translation If $k > 0$, the translation is up. If $k < 0$, the translation is down.	$(x, y) \rightarrow (x, y + k)$	
$y = f(x - h)$	A horizontal translation If $h > 0$, the translation is to the right. If $h < 0$, the translation is to the left.	$(x, y) \rightarrow (x + h, y)$	

- A sketch of the graph of $y - k = f(x - h)$, or $y = f(x - h) + k$, can be created by translating key points on the graph of the base function $y = f(x)$.

Check Your Understanding

Practise

1. For each function, state the values of h and k, the parameters that represent the horizontal and vertical translations applied to $y = f(x)$.

a) $y - 5 = f(x)$

b) $y = f(x) - 4$

c) $y = f(x + 1)$

d) $y + 3 = f(x - 7)$

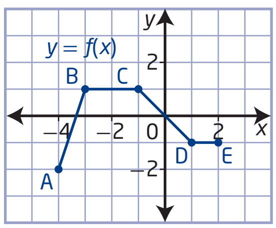
e) $y = f(x + 2) + 4$

2. Given the graph of $y = f(x)$ and each of the following transformations,

- state the coordinates of the image points A′, B′, C′, D′ and E′
- sketch the graph of the transformed function

a) $g(x) = f(x) + 3$ **b)** $h(x) = f(x - 2)$

c) $s(x) = f(x + 4)$ **d)** $t(x) = f(x) - 2$



3. Describe, using mapping notation, how the graphs of the following functions can be obtained from the graph of $y = f(x)$.

a) $y = f(x + 10)$

b) $y + 6 = f(x)$

c) $y = f(x - 7) + 4$

d) $y - 3 = f(x - 1)$

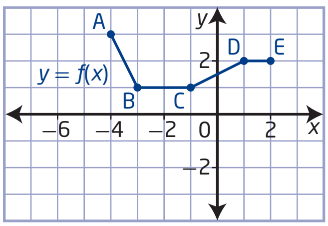
4. Given the graph of $y = f(x)$, sketch the graph of the transformed function. Describe the transformation that can be applied to the graph of $f(x)$ to obtain the graph of the transformed function. Then, write the transformation using mapping notation.

a) $r(x) = f(x + 4) - 3$

b) $s(x) = f(x - 2) - 4$

c) $t(x) = f(x - 2) + 5$

d) $v(x) = f(x + 3) + 2$



Apply

5. For each transformation, identify the values of h and k. Then, write the equation of the transformed function in the form $y - k = f(x - h)$.

a) $f(x) = \frac{1}{x}$, translated 5 units to the left and 4 units up

b) $f(x) = x^2$, translated 8 units to the right and 6 units up

c) $f(x) = |x|$, translated 10 units to the right and 8 units down

d) $y = f(x)$, translated 7 units to the left and 12 units down

6. What vertical translation is applied to $y = x^2$ if the transformed graph passes through the point (4, 19)?

7. What horizontal translation is applied to $y = x^2$ if the translation image graph passes through the point (5, 16)?

8. Copy and complete the table.

Translation	Transformed Function	Transformation of Points
vertical	$y = f(x) + 5$	$(x, y) \rightarrow (x, y + 5)$
	$y = f(x + 7)$	$(x, y) \rightarrow (x - 7, y)$
	$y = f(x - 3)$	
	$y = f(x) - 6$	
horizontal and vertical	$y + 9 = f(x + 4)$	
horizontal and vertical		$(x, y) \rightarrow (x + 4, y - 6)$
		$(x, y) \rightarrow (x - 2, y + 3)$
horizontal and vertical	$y = f(x - h) + k$	

9. The graph of the function $y = x^2$ is translated 4 units to the left and 5 units up to form the transformed function $y = g(x)$.

a) Determine the equation of the function $y = g(x)$.

b) What are the domain and range of the image function?

c) How could you use the description of the translation of the function $y = x^2$ to determine the domain and range of the image function?

10. The graph of $f(x) = |x|$ is transformed to the graph of $g(x) = f(x - 9) + 5$.

a) Determine the equation of the function $g(x)$.

b) Compare the graph of $g(x)$ to the graph of the base function $f(x)$.

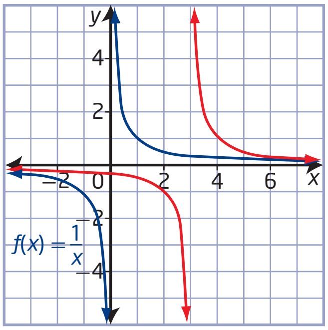
c) Determine three points on the graph of $f(x)$. Write the coordinates of the image points if you perform the horizontal translation first and then the vertical translation.

d) Using the same original points from part c), write the coordinates of the image points if you perform the vertical translation first and then the horizontal translation.

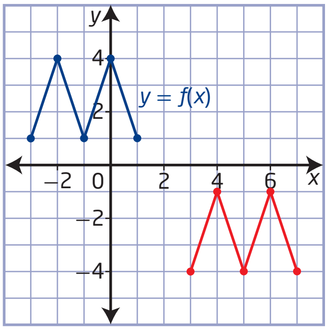
e) What do you notice about the coordinates of the image points from parts c) and d)? Is the order of the translations important?

11. The graph of the function drawn in red is a translation of the original function drawn in blue. Write the equation of the translated function in the form $y - k = f(x - h)$.

a)



b)

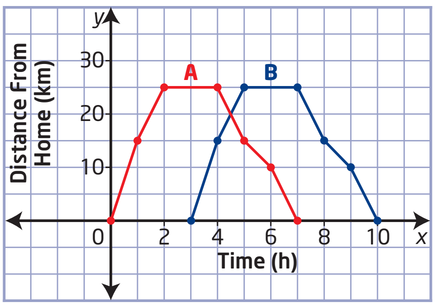


12. Janine is an avid cyclist. After cycling to a lake and back home, she graphs her distance versus time (graph A).

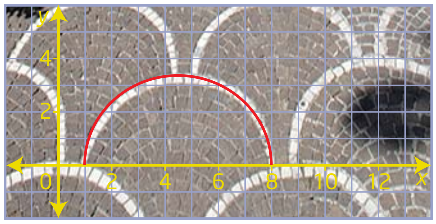
a) If she left her house at 12 noon, briefly describe a possible scenario for Janine's trip.

b) Describe the differences it would make to Janine's cycling trip if the graph of the function were translated, as shown in graph B.

c) The equation for graph A could be written as $y = f(x)$. Write the equation for graph B.



13. Architects and designers often use translations in their designs. The image shown is from an Italian roadway.



a) Use the coordinate plane overlay with the base semicircle shown to describe the approximate transformations of the semicircles.

b) If the semicircle at the bottom left of the image is defined by the function $y = f(x)$, state the approximate equations of three other semicircles.

14. This Pow Wow belt shows a frieze pattern where a particular image has been translated throughout the length of the belt.

a) With or without technology, create a design using a pattern that is a function. Use a minimum of four horizontal translations of your function to create your own frieze pattern.

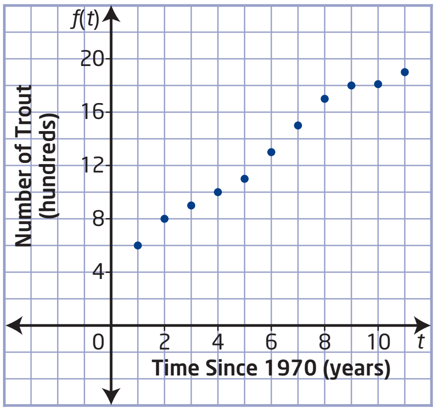
b) Describe the translation of your design in words and in an equation of the form $y = f(x - h)$.

Did You Know?

In First Nations communities today, Pow Wows have evolved into multi-tribal festivals. Traditional dances are performed by men, women, and children. The dancers wear traditional regalia specific to their dance style and nation of origin.

15. Michele Lake and Coral Lake, located near the Columbia Ice Fields, are the only two lakes in Alberta in which rare golden trout live.

Suppose the graph represents the number of golden trout in Michelle Lake in the years since 1970.



Let the function $f(t)$ represent the number of fish in Michelle Lake since 1970.

Describe an event or a situation for the fish population that would result in the following transformations of the graph. Then, use function notation to represent the transformation.

a) a vertical translation of 2 units up

b) a horizontal translation of 3 units to the right

16. Paul is an interior house painter. He determines that the function $n = f(A)$ gives the number of gallons, n, of paint needed to cover an area, A, in square metres. Interpret $n = f(A) + 10$ and $n = f(A + 10)$ in this context.

Extend

17. The graph of the function $y = x^2$ is translated to an image parabola with zeros 7 and 1.

a) Determine the equation of the image function.

b) Describe the translations on the graph of $y = x^2$.

c) Determine the y-intercept of the translated function.

18. Use translations to describe how the graph of $y = \frac{1}{x}$ compares to the graph of each function.

a) $y - 4 = \frac{1}{x}$

b) $y = \frac{1}{x + 2}$

c) $y - 3 = \frac{1}{x - 5}$

d) $y = \frac{1}{x + 3} - 4$

19. a) Predict the relationship between the graph of $y = x^3 - x^2$ and the graph of $y + 3 = (x - 2)^3 - (x - 2)^2$.

b) Graph each function to verify your prediction.

Create Connections

C1 The graph of the function $y = f(x)$ is transformed to the graph of $y = f(x - h) + k$.

a) Show that the order in which you apply translations does not matter. Explain why this is true.

b) How are the domain and range affected by the parameters h and k?

C2 Complete the square and explain how to transform the graph of $y = x^2$ to the graph of each function.

a) $f(x) = x^2 + 2x + 1$

b) $g(x) = x^2 - 4x + 3$

C3 The roots of the quadratic equation $x^2 - x - 12 = 0$ are -3 and 4. Determine the roots of the equation $(x - 5)^2 - (x - 5) - 12 = 0$.

C4 The function $f(x) = x + 4$ could be a vertical translation of 4 units up or a horizontal translation of 4 units to the left. Explain why.

Reflections and Stretches

Focus on...

- developing an understanding of the effects of reflections on the graphs of functions and their related equations
- developing an understanding of the effects of vertical and horizontal stretches on the graphs of functions and their related equations

Reflections, symmetry, as well as horizontal and vertical stretches, appear in architecture, textiles, science, and works of art. When something is symmetrical or stretched in the geometric sense, its parts have a one-to-one correspondence. How does this relate to the study of functions?

Ndebele artist, South Africa

Investigate Reflections and Stretches of Functions

Materials

- grid paper
- graphing technology

A: Graph Reflections in the *x*-Axis and the *y*-Axis

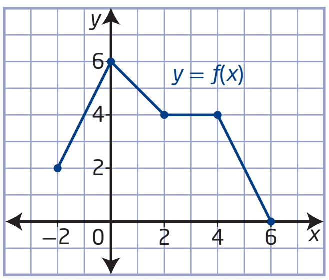
1. a) Draw a set of coordinate axes on grid paper. In quadrant I, plot a point A. Label point A with its coordinates.

b) Use the x-axis as a mirror line, or line of reflection, and plot point A′, the mirror image of point A in the x-axis.

c) How are the coordinates of points A and A′ related?

d) If point A is initially located in any of the other quadrants, does the relationship in part c) still hold true?

2. Consider the graph of the function $y = f(x)$.



a) Explain how you could graph the mirror image of the function in the x-axis.

b) Make a conjecture about how the equation of $f(x)$ changes to graph the mirror image.

3. Use graphing technology to graph the function $y = x^2 + 2x$, $-5 \leq x \leq 5$, and its mirror image in the x-axis. What equation did you enter to graph the mirror image?

4. Repeat steps 1 to 3 for a mirror image in the y-axis.

Reflect and Respond

5. Copy and complete the table to record your observations. Write concluding statements summarizing the effects of reflections in the axes.

	Reflection in	Verbal Description	Mapping	Equation of Transformed Function
Function $y = f(x)$	x-axis		$(x, y) \rightarrow (\quad , \quad)$	
	y-axis		$(x, y) \rightarrow (\quad , \quad)$	

B: Graph Vertical and Horizontal Stretches

6. a) Plot a point A on a coordinate grid and label it with its coordinates.

   b) Plot and label a point A′ with the same x-coordinate as point A, but with the y-coordinate equal to 2 times the y-coordinate of point A.

   c) Plot and label a point A″ with the same x-coordinate as point A, but with the y-coordinate equal to $\frac{1}{2}$ the y-coordinate of point A.

   d) Compare the location of points A′ and A″ to the location of the original point A. Describe how multiplying the y-coordinate by a factor of 2 or a factor of $\frac{1}{2}$ affects the position of the image point.

   Has the distance to the x-axis or the y-axis changed?

7. Consider the graph of the function $y = f(x)$ in step 2. Sketch the graph of the function when the y-values have been

   a) multiplied by 2

   b) multiplied by $\frac{1}{2}$

8. What are the equations of the transformed functions in step 7 in the form $y = af(x)$?

9. For step 7a), the graph has been vertically stretched about the x-axis by a factor of 2. Explain the statement. How would you describe the graph in step 7b)?

10. Consider the graph of the function $y = f(x)$ in step 2.

    a) If the x-values were multiplied by 2 or multiplied by $\frac{1}{2}$, describe what would happen to the graph of the function $y = f(x)$.

    b) Determine the equations of the transformed functions in part a) in the form $y = f(bx)$.

Reflect and Respond

11. Copy and complete the table to record your observations. Write concluding statements summarizing the effects of stretches about the axes.

	Stretch About	Verbal Description	Mapping	Equation of Transformed Function
Function $y = f(x)$	x-axis		$(x, y) \rightarrow ($, $)$	
	y-axis		$(x, y) \rightarrow ($, $)$	

Link the Ideas

reflection
- a transformation where each point of the original graph has an image point resulting from a reflection in a line
- may result in a change of orientation of a graph while preserving its shape

A **reflection** of a graph creates a mirror image in a line called the line of reflection. Reflections, like translations, do not change the shape of the graph. However, unlike translations, reflections may change the orientation of the graph.

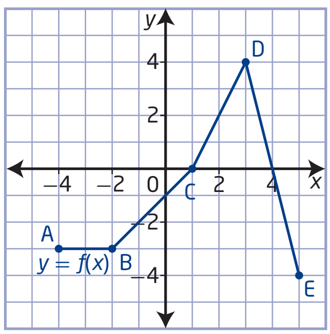
- When the output of a function $y = f(x)$ is multiplied by -1, the result, $y = -f(x)$, is a reflection of the graph in the x-axis.
- When the input of a function $y = f(x)$ is multiplied by -1, the result, $y = f(-x)$, is a reflection of the graph in the y-axis.

Example 1

Compare the Graphs of $y = f(x)$, $y = -f(x)$, and $y = f(-x)$

a) Given the graph of $y = f(x)$, graph the functions $y = -f(x)$ and $y = f(-x)$.

b) How are the graphs of $y = -f(x)$ and $y = f(-x)$ related to the graph of $y = f(x)$?



Solution

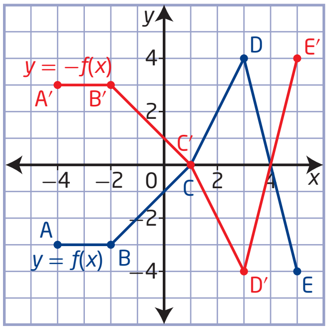
a) Use key points on the graph of $y = f(x)$ to create tables of values.

- The image points on the graph of $y = -f(x)$ have the same x-coordinates but different y-coordinates. Multiply the y-coordinates of points on the graph of $y = f(x)$ by -1.

The negative sign can be interpreted as a change in sign of one of the coordinates.

	x	$y = f(x)$
A	−4	−3
B	−2	−3
C	1	0
D	3	4
E	5	−4

	x	$y = -f(x)$
A′	−4	−1(−3) = 3
B′	−2	−1(−3) = 3
C′	1	−1(0) = 0
D′	3	−1(4) = −4
E′	5	−1(−4) = 4

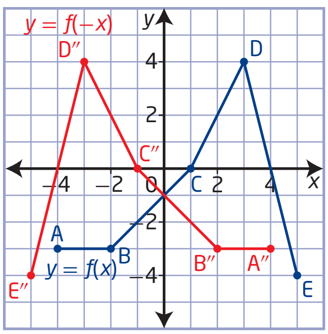


Each image point is the same distance from the line of reflection as the corresponding key point. A line drawn perpendicular to the line of reflection contains both the key point and its image point.

- The image points on the graph of $y = f(-x)$ have the same y-coordinates but different x-coordinates. Multiply the x-coordinates of points on the graph of $y = f(x)$ by -1.

	x	$y = f(x)$
A	−4	−3
B	−2	−3
C	1	0
D	3	4
E	5	−4

	x	$y = f(-x)$
A″	−1(−4) = 4	−3
B″	−1(−2) = 2	−3
C″	−1(1) = −1	0
D″	−1(3) = −3	4
E″	−1(5) = −5	−4



b) The transformed graphs are congruent to the graph of $y = f(x)$.

The points on the graph of $y = f(x)$ relate to the points on the graph of $y = -f(x)$ by the mapping $(x, y) \rightarrow (x, -y)$. The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.

Notice that the point C(1, 0) maps to itself, C′(1, 0). This point is an **invariant point**.

What is another invariant point?

invariant point

- a point on a graph that remains unchanged after a transformation is applied to it
- any point on a curve that lies on the line of reflection is an invariant point

The points on the graph of $y = f(x)$ relate to the points on the graph of $y = f(-x)$ by the mapping $(x, y) \rightarrow (-x, y)$. The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the y-axis.

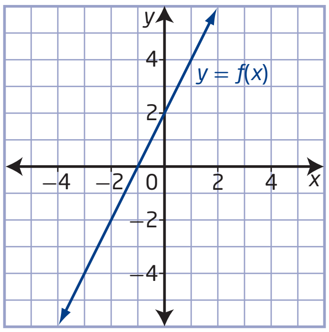
The point $(0, -1)$ is an invariant point.

Your Turn

a) Given the graph of $y = f(x)$, graph the functions $y = -f(x)$ and $y = f(-x)$.

b) Show the mapping of key points on the graph of $y = f(x)$ to image points on the graphs of $y = -f(x)$ and $y = f(-x)$.

c) Describe how the graphs of $y = -f(x)$ and $y = f(-x)$ are related to the graph of $y = f(x)$. State any invariant points.



Vertical and Horizontal Stretches

A **stretch**, unlike a translation or a reflection, changes the shape of the graph. However, like translations, stretches do not change the orientation of the graph.

stretch

- a transformation in which the distance of each x-coordinate or y-coordinate from the line of reflection is multiplied by some scale factor
- scale factors between 0 and 1 result in the point moving closer to the line of reflection; scale factors greater than 1 result in the point moving farther away from the line of reflection

- When the output of a function $y = f(x)$ is multiplied by a non-zero constant a, the result, $y = af(x)$ or $\frac{y}{a} = f(x)$, is a vertical stretch of the graph about the x-axis by a factor of $|a|$. If $a < 0$, then the graph is also reflected in the x-axis.
- When the input of a function $y = f(x)$ is multiplied by a non-zero constant b, the result, $y = f(bx)$, is a horizontal stretch of the graph about the y-axis by a factor of $\frac{1}{|b|}$. If $b < 0$, then the graph is also reflected in the y-axis.

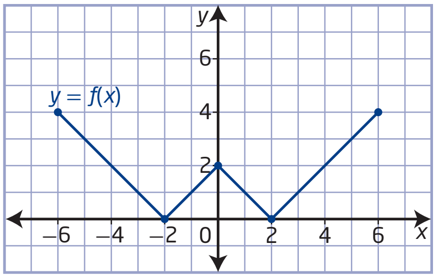
Example 2

Graph $y = af(x)$

Given the graph of $y = f(x)$,
- transform the graph of $f(x)$ to sketch the graph of $g(x)$
- describe the transformation
- state any invariant points
- state the domain and range of the functions

a) $g(x) = 2f(x)$

b) $g(x) = \frac{1}{2}f(x)$



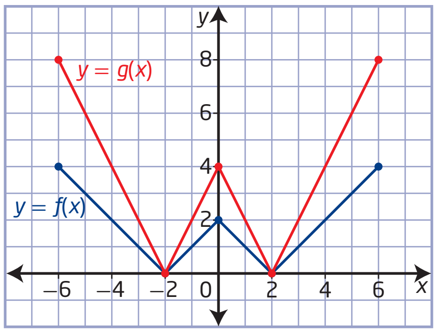
Solution

a) Use key points on the graph of $y = f(x)$ to create a table of values.

The image points on the graph of $g(x) = 2f(x)$ have the same x-coordinates but different y-coordinates. Multiply the y-coordinates of points on the graph of $y = f(x)$ by 2.

x	$y = f(x)$	$y = g(x) = 2f(x)$
–6	4	8
–2	0	0
0	2	4
2	0	0
6	4	8

The vertical distances of the transformed graph have been changed by a factor of a, where $|a| > 1$. The points on the graph of $y = af(x)$ are farther away from the x-axis than the corresponding points of the graph of $y = f(x)$.



Since $a = 2$, the points on the graph of $y = g(x)$ relate to the points on the graph of $y = f(x)$ by the mapping $(x, y) \rightarrow (x, 2y)$. Therefore, each point on the graph of $g(x)$ is twice as far from the x-axis as the corresponding point on the graph of $f(x)$. The graph of $g(x) = 2f(x)$ is a vertical stretch of the graph of $y = f(x)$ about the x-axis by a factor of 2.

The invariant points are $(-2, 0)$ and $(2, 0)$.

What is unique about the invariant points?

For $f(x)$, the domain is $\{x \mid -6 \leq x \leq 6, x \in \mathrm{R}\}$, or $[-6, 6]$, and the range is $\{y \mid 0 \leq y \leq 4, y \in \mathrm{R}\}$, or $[0, 4]$.

How can you determine the range of the new function, $g(x)$, using the range of $f(x)$ and the parameter a?

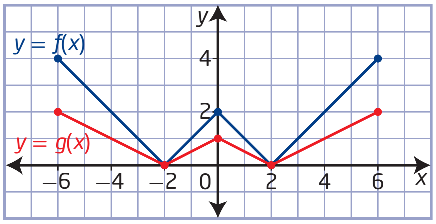
For $g(x)$, the domain is $\{x \mid -6 \leq x \leq 6, x \in \mathrm{R}\}$, or $[-6, 6]$, and the range is $\{y \mid 0 \leq y \leq 8, y \in \mathrm{R}\}$, or $[0, 8]$.

Did You Know?

There are several ways to express the domain and range of a function. For example, you can use words, a number line, set notation, or interval notation.

b) The image points on the graph of $g(x) = \frac{1}{2}f(x)$ have the same x-coordinates but different y-coordinates. Multiply the y-coordinates of points on the graph of $y = f(x)$ by $\frac{1}{2}$.

x	$y = f(x)$	$y = g(x) = \frac{1}{2}f(x)$
−6	4	2
−2	0	0
0	2	1
2	0	0
6	4	2



The vertical distances of the transformed graph have been changed by a factor a, where $0 < |a| < 1$. The points on the graph of $y = af(x)$ are closer to the x-axis than the corresponding points of the graph of $y = f(x)$.

Did You Know?

Translations and reflections are called *rigid* transformations because the shape of the graph does not change. Stretches are called *non-rigid* because the shape of the graph can change.

Since $a = \frac{1}{2}$, the points on the graph of $y = g(x)$ relate to the points on the graph of $y = f(x)$ by the mapping $(x, y) \rightarrow \left(x, \frac{1}{2}y\right)$. Therefore, each point on the graph of $g(x)$ is one half as far from the x-axis as the corresponding point on the graph of $f(x)$. The graph of $g(x) = \frac{1}{2}f(x)$ is a vertical stretch of the graph of $y = f(x)$ about the x-axis by a factor of $\frac{1}{2}$.

The invariant points are $(-2, 0)$ and $(2, 0)$.

What conclusion can you make about the invariant points after a vertical stretch?

For $f(x)$, the domain is $\{x \mid -6 \leq x \leq 6, x \in \mathrm{R}\}$, or $[-6, 6]$, and the range is $\{y \mid 0 \leq y \leq 4, y \in \mathrm{R}\}$, or $[0, 4]$.

For $g(x)$, the domain is $\{x \mid -6 \leq x \leq 6, x \in \mathrm{R}\}$, or $[-6, 6]$, and the range is $\{y \mid 0 \leq y \leq 2, y \in \mathrm{R}\}$, or $[0, 2]$.

Your Turn

Given the function $f(x) = x^2$,

- transform the graph of $f(x)$ to sketch the graph of $g(x)$
- describe the transformation
- state any invariant points
- state the domain and range of the functions

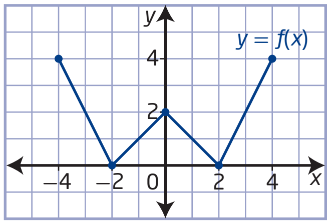
a) $g(x) = 4f(x)$

b) $g(x) = \frac{1}{3}f(x)$

Example 3

Graph $y = f(bx)$

Given the graph of $y = f(x)$,
- transform the graph of $f(x)$ to sketch the graph of $g(x)$
- describe the transformation
- state any invariant points
- state the domain and range of the functions



a) $g(x) = f(2x)$

b) $g(x) = f\left(\frac{1}{2}x\right)$

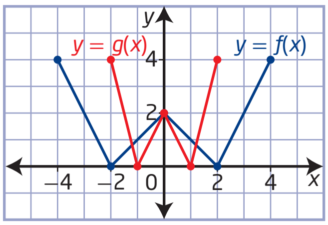
Solution

a) Use key points on the graph of $y = f(x)$ to create a table of values.

The image points on the graph of $g(x) = f(2x)$ have the same y-coordinates but different x-coordinates. Multiply the x-coordinates of points on the graph of $y = f(x)$ by $\frac{1}{2}$.

x	$y = f(x)$
−4	4
−2	0
0	2
2	0
4	4

x	$y = g(x) = f(2x)$
−2	4
−1	0
0	2
1	0
2	4



The horizontal distances of the transformed graph have been changed by a factor of $\frac{1}{b}$, where $|b| > 1$. The points on the graph of $y = f(bx)$ are closer to the y-axis than the corresponding points of the graph of $y = f(x)$.

Since $b = 2$, the points on the graph of $y = g(x)$ relate to the points on the graph of $y = f(x)$ by the mapping $(x, y) \rightarrow \left(\frac{1}{2}x, y\right)$. Therefore, each point on the graph of $g(x)$ is one half as far from the y-axis as the corresponding point on the graph of $f(x)$. The graph of $g(x) = f(2x)$ is a horizontal stretch about the y-axis by a factor of $\frac{1}{2}$ of the graph of $f(x)$.

The invariant point is $(0, 2)$.

For $f(x)$, the domain is $\{x \mid -4 \le x \le 4, x \in \mathrm{R}\}$, or $[-4, 4]$, and the range is $\{y \mid 0 \le y \le 4, y \in \mathrm{R}\}$, or $[0, 4]$.

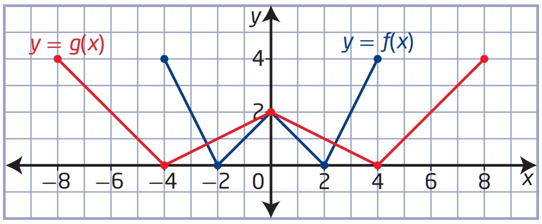
For $g(x)$, the domain is $\{x \mid -2 \le x \le 2, x \in \mathrm{R}\}$, or $[-2, 2]$, and the range is $\{y \mid 0 \le y \le 4, y \in \mathrm{R}\}$, or $[0, 4]$.

How can you determine the domain of the new function, $g(x)$, using the domain of $f(x)$ and the parameter b?

b) The image points on the graph of $g(x) = f\left(\frac{1}{2}x\right)$ have the same y-coordinates but different x-coordinates. Multiply the x-coordinates of points on the graph of $y = f(x)$ by 2.

x	$y = f(x)$
−4	4
−2	0
0	2
2	0
4	4

x	$y = g(x) = f\left(\frac{1}{2}x\right)$
−8	4
−4	0
0	2
4	0
8	4



The horizontal distances of the transformed graph have been changed by a factor $\frac{1}{b}$, where $0 < |b| < 1$. The points on the graph of $y = f(bx)$ are farther away from the y-axis than the corresponding points of the graph of $y = f(x)$.

Since $b = \frac{1}{2}$, the points on the graph of $y = g(x)$ relate to the points on the graph of $y = f(x)$ by the mapping $(x, y) \rightarrow (2x, y)$. Therefore, each point on the graph of $g(x)$ is twice as far from the y-axis as the corresponding point on the graph of $f(x)$. The graph of $g(x) = f\left(\frac{1}{2}x\right)$ is a horizontal stretch about the y-axis by a factor of 2 of the graph of $f(x)$.

The invariant point is $(0, 2)$.

How do you know which points will be invariant points after a horizontal stretch?

For $f(x)$, the domain is $\{x \mid -4 \le x \le 4, x \in \mathrm{R}\}$, or $[-4, 4]$, and the range is $\{y \mid 0 \le y \le 4, y \in \mathrm{R}\}$, or $[0, 4]$.

For $g(x)$, the domain is $\{x \mid -8 \le x \le 8, x \in \mathrm{R}\}$, or $[-8, 8]$, and the range is $\{y \mid 0 \le y \le 4, y \in \mathrm{R}\}$, or $[0, 4]$.

Your Turn

Given the function $f(x) = x^2$,
- transform the graph of $f(x)$ to sketch the graph of $g(x)$
- describe the transformation
- state any invariant points
- state the domain and range of the functions

a) $g(x) = f(3x)$

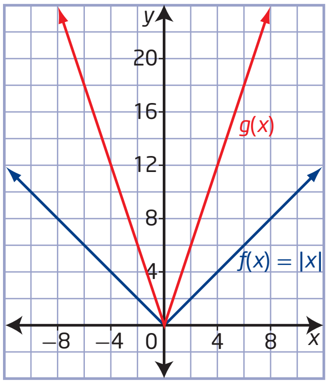
b) $g(x) = f\left(\frac{1}{4}x\right)$

Example 4

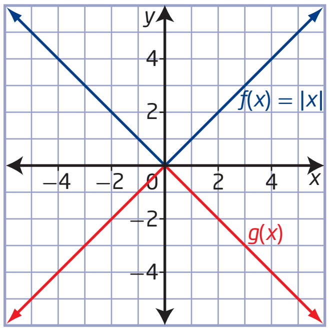
Write the Equation of a Transformed Function

The graph of the function $y = f(x)$ has been transformed by either a stretch or a reflection. Write the equation of the transformed graph, $g(x)$.

a)



b)



Solution

a) Notice that the V-shape has changed, so the graph has been transformed by a stretch.

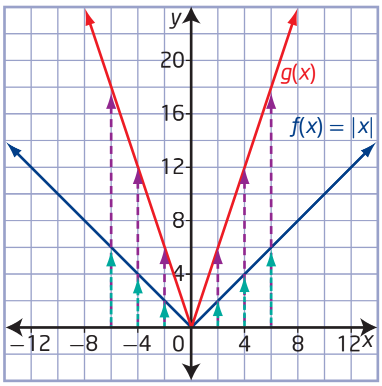
Since the original function is $f(x) = |x|$, a stretch can be described in two ways.

Why is this the case?

Choose key points on the graph of $y = f(x)$ and determine their image points on the graph of the transformed function, $g(x)$.

Case 1
Check for a pattern in the y-coordinates.

x	$y = f(x)$	$y = g(x)$
–6	6	18
–4	4	12
–2	2	6
0	0	0
2	2	6
4	4	12
6	6	18



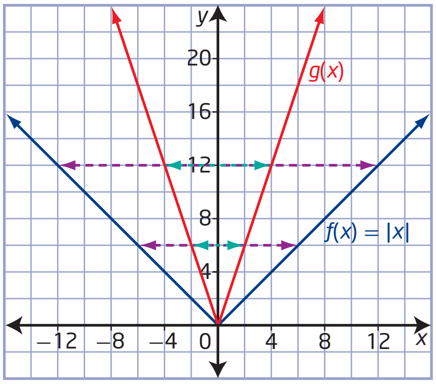
A vertical stretch results when the vertical distances of the transformed graph are a constant multiple of those of the original graph with respect to the x-axis.

The transformation can be described by the mapping $(x, y) \rightarrow (x, 3y)$. This is of the form $y = af(x)$, indicating that there is a vertical stretch about the x-axis by a factor of 3. The equation of the transformed function is $g(x) = 3f(x)$ or $g(x) = 3|x|$.

Case 2
Check for a pattern in the x-coordinates.

x	$y = f(x)$
−12	12
−6	6
0	0
6	6
12	12

x	$y = g(x)$
−4	12
−2	6
0	0
2	6
4	12



A horizontal stretch results when the horizontal distances of the transformed graph are a constant multiple of those of the original graph with respect to the y-axis.

The transformation can be described by the mapping $(x, y) \rightarrow \left(\frac{1}{3}x, y\right)$. This is of the form $y = f(bx)$, indicating that there is a horizontal stretch about the y-axis by a factor of $\frac{1}{3}$. The equation of the transformed function is $g(x) = f(3x)$ or $g(x) = |3x|$.

b) Notice that the shape of the graph has not changed, so the graph has been transformed by a reflection.

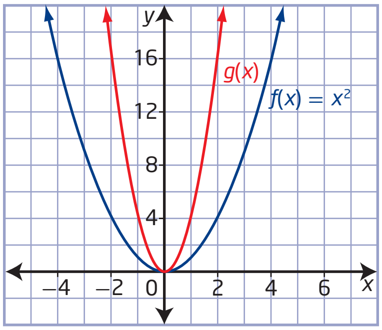
Choose key points on the graph of $f(x) = |x|$ and determine their image points on the graph of the transformed function, $g(x)$.

x	$y = f(x)$	$y = g(x)$
−4	4	−4
−2	2	−2
0	0	0
2	2	−2
4	4	−4

The transformation can be described by the mapping $(x, y) \rightarrow (x, -y)$. This is of the form $y = -f(x)$, indicating a reflection in the x-axis. The equation of the transformed function is $g(x) = -|x|$.

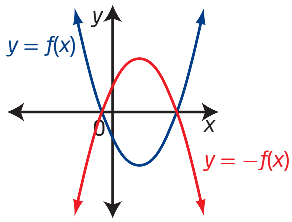
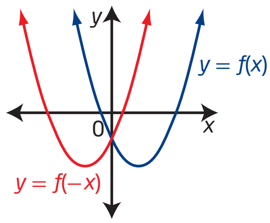
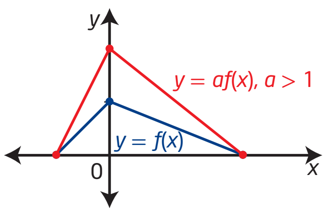
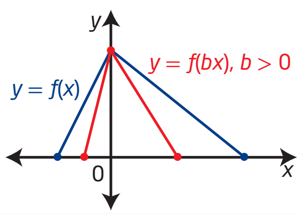
Your Turn

The graph of the function $y = f(x)$ has been transformed. Write the equation of the transformed graph, $g(x)$.



Key Ideas

- Any point on a line of reflection is an invariant point.

Function	Transformation from $y = f(x)$	Mapping	Example
$y = -f(x)$	A reflection in the x-axis	$(x, y) \rightarrow (x, -y)$	
$y = f(-x)$	A reflection in the y-axis	$(x, y) \rightarrow (-x, y)$	
$y = af(x)$	A vertical stretch about the x-axis by a factor of $\lvert a \rvert$; if $a < 0$, then the graph is also reflected in the x-axis	$(x, y) \rightarrow (x, ay)$	
$y = f(bx)$	A horizontal stretch about the y-axis by a factor of $\frac{1}{\lvert b \rvert}$; if $b < 0$, then the graph is also reflected in the y-axis	$(x, y) \rightarrow \left(\frac{x}{b}, y\right)$	

Check Your Understanding

Practise

1. a) Copy and complete the table of values for the given functions.

x	$f(x) = 2x + 1$	$g(x) = -f(x)$	$h(x) = f(-x)$
-4			
-2			
0			
2			
4			

   b) Sketch the graphs of $f(x)$, $g(x)$, and $h(x)$ on the same set of coordinate axes.

   c) Explain how the points on the graphs of $g(x)$ and $h(x)$ relate to the transformation of the function $f(x) = 2x + 1$. List any invariant points.

   d) How is each function related to the graph of $f(x) = 2x + 1$?

2. a) Copy and complete the table of values for the given functions.

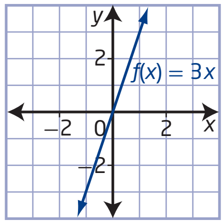
x	$f(x) = x^2$	$g(x) = 3f(x)$	$h(x) = \frac{1}{3}f(x)$
-6			
-3			
0			
3			
6			

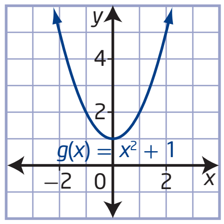
   b) Sketch the graphs of $f(x)$, $g(x)$, and $h(x)$ on the same set of coordinate axes.

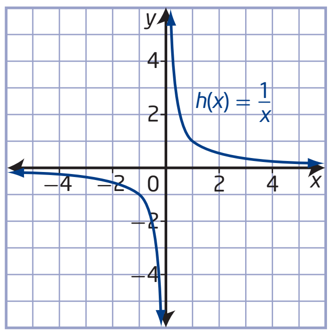
   c) Explain how the points on the graphs of $g(x)$ and $h(x)$ relate to the transformation of the function $f(x) = x^2$. List any invariant points.

   d) How is each function related to the graph of $f(x) = x^2$?

3. Consider each graph of a function.
   - Copy the graph of the function and sketch its reflection in the x-axis on the same set of axes.
   - State the equation of the reflected function in simplified form.
   - State the domain and range of each function.

   a)


   b)


   c)


4. Consider each function in #3.
   - Copy the graph of the function and sketch its reflection in the y-axis on the same set of axes.
   - State the equation of the reflected function.
   - State the domain and range for each function.

5. Use words and mapping notation to describe how the graph of each function can be found from the graph of the function $y = f(x)$.

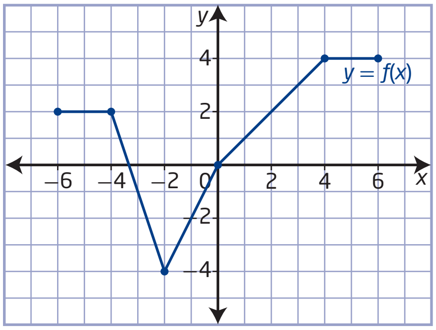
a) $y = 4f(x)$

b) $y = f(3x)$

c) $y = -f(x)$

d) $y = f(-x)$

6. The graph of the function $y = f(x)$ is vertically stretched about the x-axis by a factor of 2.

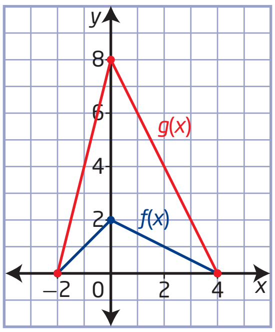


a) Determine the domain and range of the transformed function.

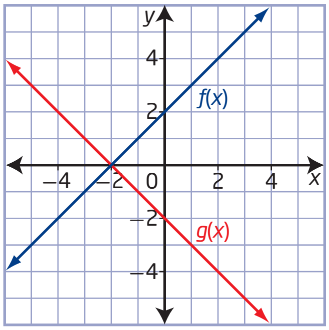
b) Explain the effect that a vertical stretch has on the domain and range of a function.

7. Describe the transformation that must be applied to the graph of $f(x)$ to obtain the graph of $g(x)$. Then, determine the equation of $g(x)$ in the form $y = af(bx)$.

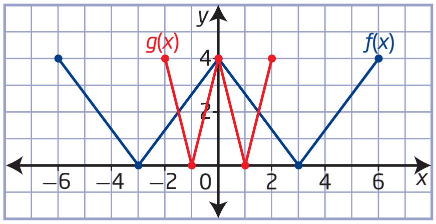
a)



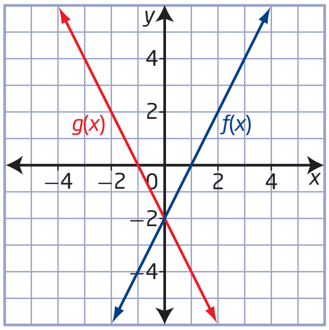
b)



c)

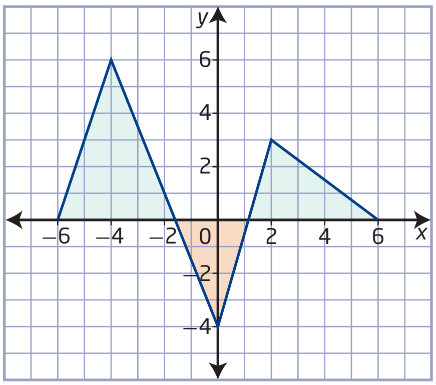


d)



Apply

8. A weaver sets up a pattern on a computer using the graph shown. A new line of merchandise calls for the design to be altered to $y = f(0.5x)$. Sketch the graph of the new design.



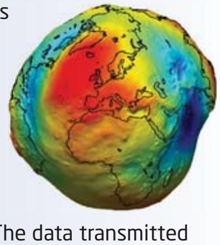
9. Describe what happens to the graph of a function $y = f(x)$ after the following changes are made to its equation.
   a) Replace x with $4x$.
   b) Replace x with $\frac{1}{4}x$.
   c) Replace y with $2y$.
   d) Replace y with $\frac{1}{4}y$.
   e) Replace x with $-3x$.
   f) Replace y with $-\frac{1}{3}y$.

10. Thomas and Sharyn discuss the order of the transformations of the graph of $y = -3|x|$ compared to the graph of $y = |x|$. Thomas states that the reflection must be applied first. Sharyn claims that the vertical stretch should be applied first.
   a) Sketch the graph of $y = -3|x|$ by applying the reflection first.
   b) Sketch the graph of $y = -3|x|$ by applying the stretch first.
   c) Explain your conclusions. Who is correct?

11. An object falling in a vacuum is affected only by the gravitational force. An equation that can model a free-falling object on Earth is $d = -4.9t^2$, where d is the distance travelled, in metres, and t is the time, in seconds. An object free falling on the moon can be modelled by the equation $d = -1.6t^2$.
   a) Sketch the graph of each function.
   b) Compare each function equation to the base function $d = t^2$.

Did You Know?

The actual strength of Earth's gravity varies depending on location.



On March 17, 2009, the European Space Agency launched a gravity-mapping satellite called Gravity and Ocean Circulation Explorer (GOCE). The data transmitted from GOCE are being used to build a model of Earth's shape and a gravity map of the planet.

12. Explain the differences that occur in transforming the graph of the function $y = f(x)$ to the graph of the function $y = f(bx)$ as compared to transforming $y = f(x)$ to $y = af(x)$.

13. The speed of a vehicle the moment the brakes are applied can be determined by its skid marks. The length, D, in feet, of the skid mark is related to the speed, S, in miles per hour, of the vehicle before braking by the function $D = \frac{1}{30fn}S^2$, where f is the drag factor of the road surface and n is the braking efficiency as a decimal. Suppose the braking efficiency is 100% or 1.
   a) Sketch the graph of the length of the skid mark as a function of speed for a drag factor of 1, or $D = \frac{1}{30}S^2$.
   b) The drag factor for asphalt is 0.9, for gravel is 0.8, for snow is 0.55, and for ice is 0.25. Compare the graphs of the functions for these drag factors to the graph in part a).

Did You Know?

A technical accident investigator or reconstructionist is a specially trained police officer who investigates serious traffic accidents. These officers use photography, measurements of skid patterns, and other information to determine the cause of the collision and if any charges should be laid.

Extend

14. Consider the function $f(x) = (x + 4)(x - 3)$. Without graphing, determine the zeros of the function after each transformation.

a) $y = 4f(x)$

b) $y = f(-x)$

c) $y = f\left(\frac{1}{2}x\right)$

d) $y = f(2x)$

15. The graph of a function $y = f(x)$ is contained completely in the fourth quadrant. Copy and complete each statement.

a) If $y = f(x)$ is transformed to $y = -f(x)$, it will be in quadrant ■.

b) If $y = f(x)$ is transformed to $y = f(-x)$, it will be in quadrant ■.

c) If $y = f(x)$ is transformed to $y = 4f(x)$, it will be in quadrant ■.

d) If $y = f(x)$ is transformed to $y = f\left(\frac{1}{4}x\right)$, it will be in quadrant ■.

16. Sketch the graph of $f(x) = |x|$ reflected in each line.

a) $x = 3$

b) $y = -2$

Create Connections

C1 Explain why the graph of $g(x) = f(bx)$ is a horizontal stretch about the y-axis by a factor of $\frac{1}{b}$, for $b > 0$, rather than a factor of b.

C2 Describe a transformation that results in each situation. Is there more than one possibility?

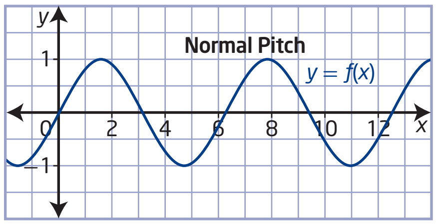
a) The x-intercepts are invariant points.

b) The y-intercepts are invariant points.

C3 A point on the function $f(x)$ is mapped onto the image point on the function $g(x)$. Copy and complete the table by describing a possible transformation of $f(x)$ to obtain $g(x)$ for each mapping.

$f(x)$	$g(x)$	Transformation
(5, 6)	(5, –6)	
(4, 8)	(–4, 8)	
(2, 3)	(2, 12)	
(4, –12)	(2, –6)	

C4 Sound is a form of energy produced and transmitted by vibrating matter that travels in waves. Pitch is the measure of how high or how low a sound is. The graph of $f(x)$ demonstrates a normal pitch. Copy the graph, then sketch the graphs of $y = f(3x)$, indicating a higher pitch, and $y = f\left(\frac{1}{2}x\right)$, for a lower pitch.



Did You Know?

The *pitch* of a sound wave is directly related to its *frequency*. A high-pitched sound has a high frequency (a mosquito). A low-pitched sound has a low frequency (a fog-horn).

A healthy human ear can hear frequencies in the range of 20 Hz to 20 000 Hz.

C5 **a)** Write the equation for the general term of the sequence $-10, -6, -2, 2, 6, \ldots$.

b) Write the equation for the general term of the sequence $10, 6, 2, -2, -6, \ldots$.

c) How are the graphs of the two sequences related?

1.3

Combining Transformations

Focus on…

- sketching the graph of a transformed function by applying translations, reflections, and stretches
- writing the equation of a function that has been transformed from the function $y = f(x)$

Architects, artists, and craftspeople use transformations in their work. Towers that stretch the limits of architectural technologies, paintings that create futuristic landscapes from ordinary objects, and quilt designs that transform a single shape to create a more complex image are examples of these transformations.

In this section, you will apply a combination of transformations to base functions to create more complex functions.

National-Nederlanden Building in Prague, Czech Republic

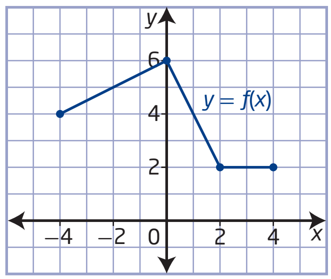
Investigate the Order of Transformations

Materials

- grid paper

New graphs can be created by vertical or horizontal translations, vertical or horizontal stretches, or reflections in an axis. When vertical and horizontal translations are applied to the graph of a function, the order in which they occur does not affect the position of the final image.

Explore whether order matters when other combinations of transformations are applied. Consider the graph of $y = f(x)$.



A: Stretches

1. a) Copy the graph of $y = f(x)$.

b) Sketch the transformed graph after the following two stretches are performed in order. Write the resulting function equation after each transformation.

- Stretch vertically about the x-axis by a factor of 2.
- Stretch horizontally about the y-axis by a factor of 3.

c) Sketch the transformed graph after the same two stretches are performed in reverse order. Write the resulting function equation after each transformation.
- Stretch horizontally about the y-axis by a factor of 3.
- Stretch vertically about the x-axis by a factor of 2.

2. Compare the final graphs and equations from step 1b) and c). Did reversing the order of the stretches change the final result?

B: Combining Reflections and Translations

3. a) Copy the graph of $y = f(x)$.

b) Sketch the transformed graph after the following two transformations are performed in order. Write the resulting function equation after each transformation.
- Reflect in the x-axis.
- Translate vertically 4 units up.

c) Sketch the transformed graph after the same two transformations are performed in reverse order. Write the resulting function equation after each transformation.
- Translate vertically 4 units up.
- Reflect in the x-axis.

4. Compare the final graphs and equations from step 3b) and c). Did reversing the order of the transformations change the final result? Explain.

5. a) Copy the graph of $y = f(x)$.

b) Sketch the transformed graph after the following two transformations are performed in order. Write the resulting function equation after each transformation.
- Reflect in the y-axis.
- Translate horizontally 4 units to the right.

c) Sketch the transformed graph after the same two transformations are performed in reverse order. Write the resulting function equation after each transformation.
- Translate horizontally 4 units to the right.
- Reflect in the y-axis.

6. Compare the final graphs and equations from step 5b) and c). Did reversing the order of the transformations change the final result? Explain.

Reflect and Respond

7. a) What do you think would happen if the graph of a function were transformed by a vertical stretch about the x-axis and a vertical translation? Would the order of the transformations matter?

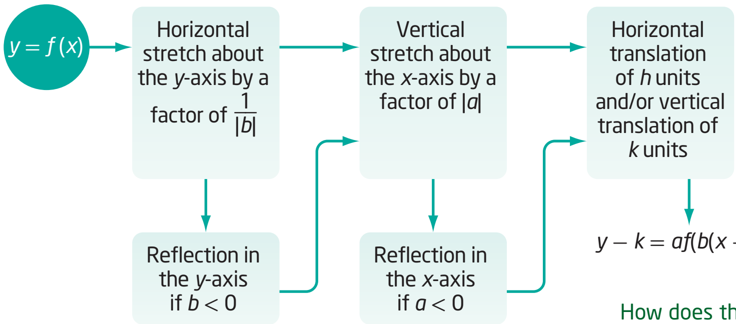
b) Use the graph of $y = |x|$ to test your prediction.

8. In which order do you think transformations should be performed to produce the correct graph? Explain.

Link the Ideas

Multiple transformations can be applied to a function using the general transformation model $y - k = af(b(x - h))$ or $y = af(b(x - h)) + k$.

To accurately sketch the graph of a function of the form $y - k = af(b(x - h))$, the stretches and reflections (values of a and b) should occur before the translations (h-value and k-value). The diagram shows one recommended sequence for the order of transformations.



How does this compare to the usual order of operations?

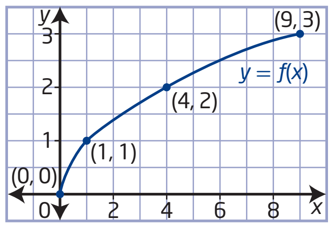
Example 1

Graph a Transformed Function

Describe the combination of transformations that must be applied to the function $y = f(x)$ to obtain the transformed function. Sketch the graph, showing each step of the transformation.

a) $y = 3f(2x)$

b) $y = f(3x + 6)$

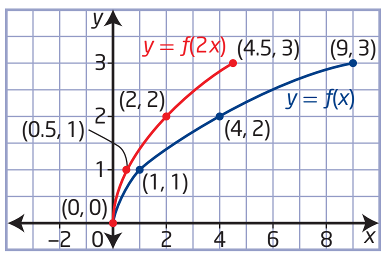


Solution

a) Compare the function to $y = af(b(x - h)) + k$. For $y = 3f(2x)$, $a = 3$, $b = 2$, $h = 0$, and $k = 0$.

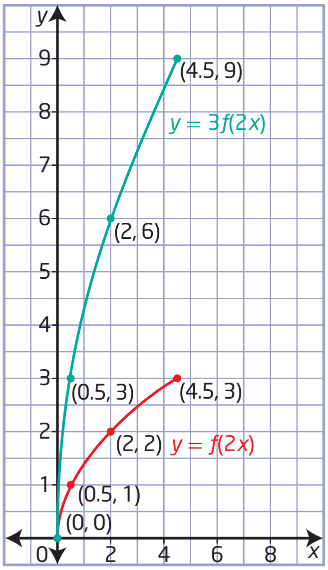
The graph of $y = f(x)$ is horizontally stretched about the y-axis by a factor of $\frac{1}{2}$ and then vertically stretched about the x-axis by a factor of 3.

- Apply the horizontal stretch by a factor of $\frac{1}{2}$ to obtain the graph of $y = f(2x)$.



- Apply the vertical stretch by a factor of 3 to $y = f(2x)$ to obtain the graph of $y = 3f(2x)$.

Would performing the stretches in reverse order change the final result?

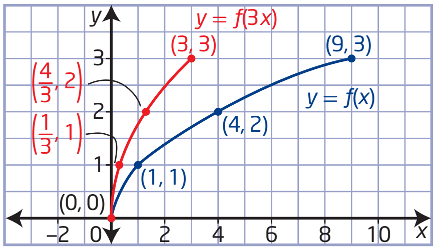


b) First, rewrite $y = f(3x + 6)$ in the form $y = af(b(x - h)) + k$. This makes it easier to identify specific transformations.

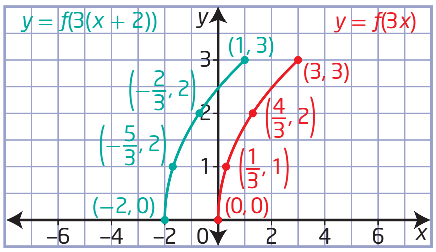
$y = f(3x + 6)$
$y = f(3(x + 2))$ Factor out the coefficient of *x*.
For $y = f(3(x + 2))$, $a = 1$, $b = 3$, $h = -2$, and $k = 0$.

The graph of $y = f(x)$ is horizontally stretched about the *y*-axis by a factor of $\frac{1}{3}$ and then horizontally translated 2 units to the left.

- Apply the horizontal stretch by a factor of $\frac{1}{3}$ to obtain the graph of $y = f(3x)$.



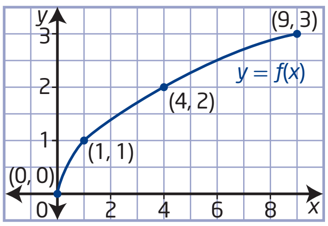
- Apply the horizontal translation of 2 units to the left to $y = f(3x)$ to obtain the graph of $y = f(3(x + 2))$.



Your Turn

Describe the combination of transformations that must be applied to the function $y = f(x)$ to obtain the transformed function. Sketch the graph, showing each step of the transformation.

a) $y = 2f(x) - 3$ **b)** $y = f\left(\frac{1}{2}x - 2\right)$



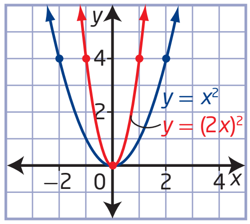
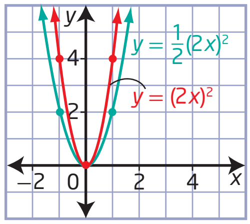
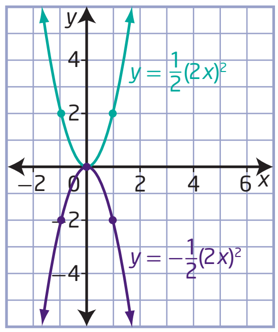
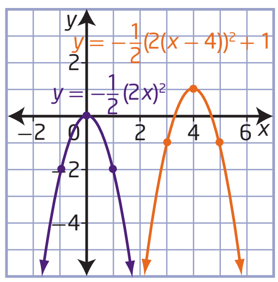
Example 2

Combination of Transformations

Show the combination of transformations that should be applied to the graph of the function $f(x) = x^2$ in order to obtain the graph of the transformed function $g(x) = -\frac{1}{2}f(2(x - 4)) + 1$. Write the corresponding equation for $g(x)$.

Solution

For $g(x) = -\frac{1}{2}f(2(x - 4)) + 1$, $a = -\frac{1}{2}$, $b = 2$, $h = 4$, and $k = 1$.

Description	Mapping	Graph
Horizontal stretch about the y-axis by a factor of $\frac{1}{2}$ $y = (2x)^2$	$(-2, 4) \rightarrow (-1, 4)$ $(0, 0) \rightarrow (0, 0)$ $(2, 4) \rightarrow (1, 4)$ $(x, y) \rightarrow \left(\frac{1}{2}x, y\right)$	
Vertical stretch about the x-axis by a factor of $\frac{1}{2}$ $y = \frac{1}{2}(2x)^2$	$(-1, 4) \rightarrow (-1, 2)$ $(0, 0) \rightarrow (0, 0)$ $(1, 4) \rightarrow (1, 2)$ $\left(\frac{1}{2}x, y\right) \rightarrow \left(\frac{1}{2}x, \frac{1}{2}y\right)$	
Reflection in the x-axis $y = -\frac{1}{2}(2x)^2$	$(-1, 2) \rightarrow (-1, -2)$ $(0, 0) \rightarrow (0, 0)$ $(1, 2) \rightarrow (1, -2)$ $\left(\frac{1}{2}x, \frac{1}{2}y\right) \rightarrow \left(\frac{1}{2}x, -\frac{1}{2}y\right)$	
Translation of 4 units to the right and 1 unit up $y = -\frac{1}{2}(2(x - 4))^2 + 1$	$(-1, -2) \rightarrow (3, -1)$ $(0, 0) \rightarrow (4, 1)$ $(1, -2) \rightarrow (5, -1)$ $\left(\frac{1}{2}x, -\frac{1}{2}y\right) \rightarrow \left(\frac{1}{2}x + 4, -\frac{1}{2}y + 1\right)$	

The equation of the transformed function is $g(x) = -\frac{1}{2}(2(x - 4))^2 + 1$.

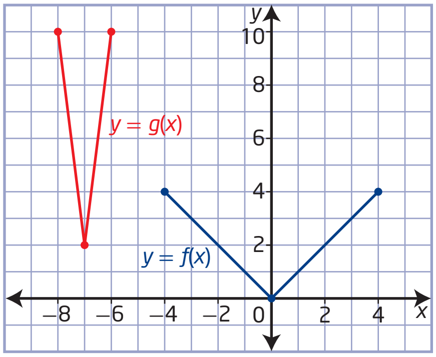
Your Turn

Describe the combination of transformations that should be applied to the function $f(x) = x^2$ in order to obtain the transformed function $g(x) = -2f\left(\frac{1}{2}(x + 8)\right) - 3$. Write the corresponding equation and sketch the graph of $g(x)$.

Example 3

Write the Equation of a Transformed Function Graph

The graph of the function $y = g(x)$ represents a transformation of the graph of $y = f(x)$. Determine the equation of $g(x)$ in the form $y = af(b(x - h)) + k$. Explain your answer.



Solution

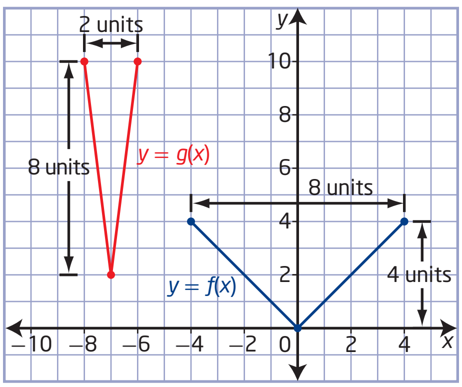
Locate key points on the graph of $f(x)$ and their image points on the graph of $g(x)$.

$(-4, 4) \rightarrow (-8, 10)$
$(0, 0) \rightarrow (-7, 2)$
$(4, 4) \rightarrow (-6, 10)$

The point (0, 0) on the graph of $f(x)$ is not affected by any stretch, either horizontal or vertical, or any reflection so it can be used to determine the vertical and horizontal translations. The graph of $g(x)$ has been translated 7 units to the left and 2 units up.

$h = -7$ and $k = 2$
There is no reflection.

Compare the distances between key points. In the vertical direction, 4 units becomes 8 units. There is a vertical stretch by a factor of 2. In the horizontal direction, 8 units becomes 2 units. There is also a horizontal stretch by a factor of $\frac{1}{4}$.



$a = 2$ and $b = 4$

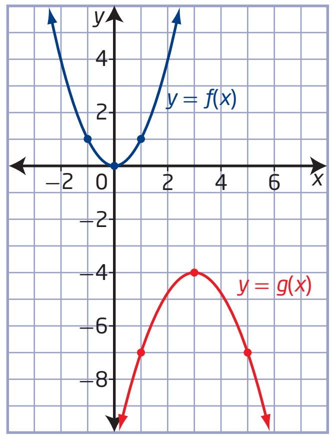
Substitute the values of a, b, h, and k into $y = af(b(x - h)) + k$.

The equation of the transformed function is $g(x) = 2f(4(x + 7)) + 2$.

How could you use the mapping $(x, y) \rightarrow \left(\frac{1}{b}x + h, ay + k\right)$ to verify this equation?

Your Turn

The graph of the function $y = g(x)$ represents a transformation of the graph of $y = f(x)$. State the equation of the transformed function. Explain your answer.



Key Ideas

- Write the function in the form $y = af(b(x - h)) + k$ to better identify the transformations.
- Stretches and reflections may be performed in any order before translations.
- The parameters a, b, h, and k in the function $y = af(b(x - h)) + k$ correspond to the following transformations:
  - a corresponds to a vertical stretch about the x-axis by a factor of $|a|$. If $a < 0$, then the function is reflected in the x-axis.
  - b corresponds to a horizontal stretch about the y-axis by a factor of $\frac{1}{|b|}$. If $b < 0$, then the function is reflected in the y-axis.
  - h corresponds to a horizontal translation.
  - k corresponds to a vertical translation.

Check Your Understanding

Practise

1. The function $y = x^2$ has been transformed to $y = af(bx)$. Determine the equation of each transformed function.

a) Its graph is stretched horizontally about the y-axis by a factor of 2 and then reflected in the x-axis.

b) Its graph is stretched horizontally about the y-axis by a factor of $\frac{1}{4}$, reflected in the y-axis, and then stretched vertically about the x-axis by a factor of $\frac{1}{4}$.

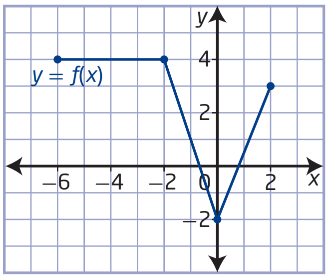
2. The function $y = f(x)$ is transformed to the function $g(x) = -3f(4x - 16) - 10$. Copy and complete the following statements by filling in the blanks.

The function $f(x)$ is transformed to the function $g(x)$ by a horizontal stretch about the ■ by a factor of ■. It is vertically stretched about the ■ by a factor of ■. It is reflected in the ■, and then translated ■ units to the right and ■ units down.

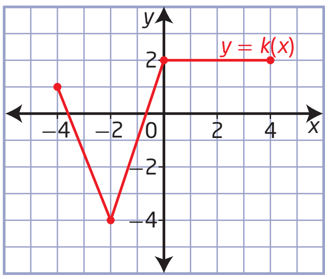
3. Copy and complete the table by describing the transformations of the given functions, compared to the function $y = f(x)$.

Function	Reflections	Vertical Stretch Factor	Horizontal Stretch Factor	Vertical Translation	Horizontal Translation
$y - 4 = f(x - 5)$					
$y + 5 = 2f(3x)$					
$y = \frac{1}{2}f\left(\frac{1}{2}(x - 4)\right)$					
$y + 2 = -3f(2(x + 2))$					

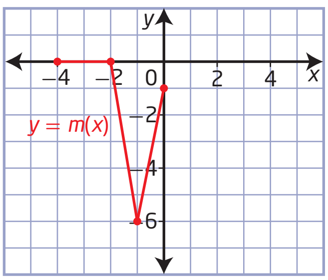
4. Using the graph of $y = f(x)$, write the equation of each transformed graph in the form $y = af(b(x - h)) + k$.



a)

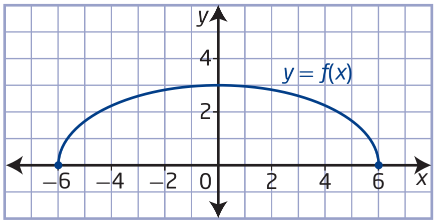


b)



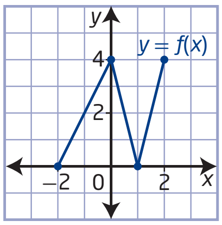
5. For each graph of $y = f(x)$, sketch the graph of the combined transformations. Show each transformation in the sequence.

a)



- vertical stretch about the x-axis by a factor of 2
- horizontal stretch about the y-axis by a factor of $\frac{1}{3}$
- translation of 5 units to the left and 3 units up

b)



- vertical stretch about the x-axis by a factor of $\frac{3}{4}$
- horizontal stretch about the y-axis by a factor of 3
- translation of 3 units to the right and 4 units down

6. The key point $(-12, 18)$ is on the graph of $y = f(x)$. What is its image point under each transformation of the graph of $f(x)$?

a) $y + 6 = f(x - 4)$

b) $y = 4f(3x)$

c) $y = -2f(x - 6) + 4$

d) $y = -2f\left(-\frac{2}{3}x - 6\right) + 4$

e) $y + 3 = -\frac{1}{3}f(2(x + 6))$

Apply

7. Describe, using an appropriate order, how to obtain the graph of each function from the graph of $y = f(x)$. Then, give the mapping for the transformation.

a) $y = 2f(x - 3) + 4$

b) $y = -f(3x) - 2$

c) $y = -\frac{1}{4}f(-(x + 2))$

d) $y - 3 = -f(4(x - 2))$

e) $y = -\frac{2}{3}f\left(-\frac{3}{4}x\right)$

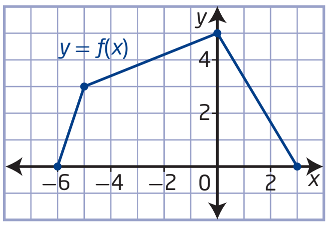
f) $3y - 6 = f(-2x + 12)$

8. Given the function $y = f(x)$, write the equation of the form $y - k = af(b(x - h))$ that would result from each combination of transformations.

a) a vertical stretch about the x-axis by a factor of 3, a reflection in the x-axis, a horizontal translation of 4 units to the left, and a vertical translation of 5 units down

b) a horizontal stretch about the y-axis by a factor of $\frac{1}{3}$, a vertical stretch about the x-axis by a factor of $\frac{3}{4}$, a reflection in both the x-axis and the y-axis, and a translation of 6 units to the right and 2 units up

9. The graph of $y = f(x)$ is given. Sketch the graph of each of the following functions.



a) $y + 2 = f(x - 3)$

b) $y = -f(-x)$

c) $y = f(3(x - 2)) + 1$

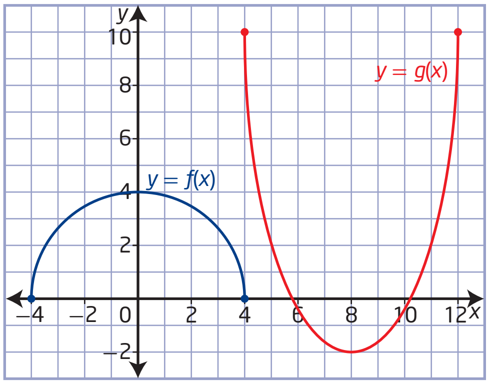
d) $y = 3f\left(\frac{1}{3}x\right)$

e) $y + 2 = -3f(x + 4)$

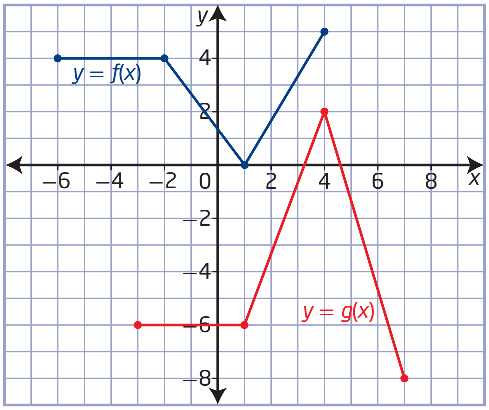
f) $y = \frac{1}{2}f\left(-\frac{1}{2}(x + 2)\right) - 1$

10. The graph of the function $y = g(x)$ represents a transformation of the graph of $y = f(x)$. Determine the equation of $g(x)$ in the form $y = af(b(x - h)) + k$.

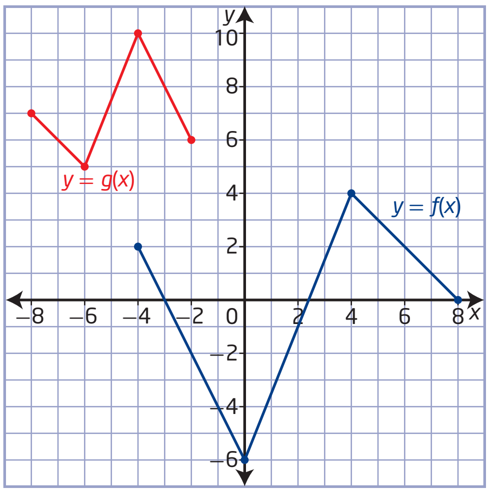
a)



b)



c)



11. Given the function $f(x)$, sketch the graph of the transformed function $g(x)$.

a) $f(x) = x^2$, $g(x) = -2f(4(x + 2)) - 2$

b) $f(x) = |x|$, $g(x) = -2f(-3x + 6) + 4$

c) $f(x) = x$, $g(x) = -\frac{1}{3}f(-2(x + 3)) - 2$

12. Alison often sketches her quilt designs on a coordinate grid. The coordinates for a section of one her designs are A(−4, 6), B(−2, −2), C(0, 0), D(1, −1), and E(3, 6). She wants to transform the original design by a horizontal stretch about the y-axis by a factor of 2, a reflection in the x-axis, and a translation of 4 units up and 3 units to the left.

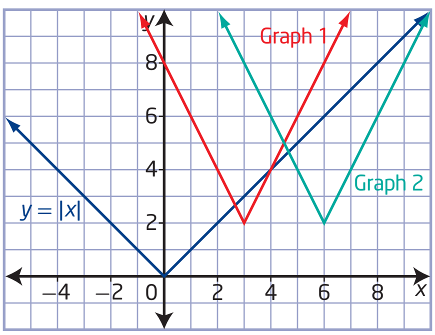
a) Determine the coordinates of the image points, A′, B′, C′, D′, and E′.

b) If the original design was defined by the function $y = f(x)$, determine the equation of the design resulting from the transformations.

13. Gil is asked to translate the graph of $y = |x|$ according to the equation $y = |2x - 6| + 2$. He decides to do the horizontal translation of 3 units to the right first, then the stretch about the y-axis by a factor of $\frac{1}{2}$, and lastly the translation of 2 units up. This gives him Graph 1. To check his work, he decides to apply the horizontal stretch about the y-axis by a factor of $\frac{1}{2}$ first, and then the horizontal translation of 6 units to the right and the vertical translation of 2 units up. This results in Graph 2.

a) Explain why the two graphs are in different locations.

b) How could Gil have rewritten the equation so that the order in which he did the transformations for Graph 2 resulted in the same position as Graph 1?



14. Two parabolic arches are being built. The first arch can be modelled by the function $y = -x^2 + 9$, with a range of $0 \le y \le 9$. The second arch must span twice the distance and be translated 6 units to the left and 3 units down.

a) Sketch the graph of both arches.

b) Determine the equation of the second arch.

Extend

15. If the x-intercept of the graph of $y = f(x)$ is located at $(a, 0)$ and the y-intercept is located at $(0, b)$, determine the x-intercept and y-intercept after the following transformations of the graph of $y = f(x)$.

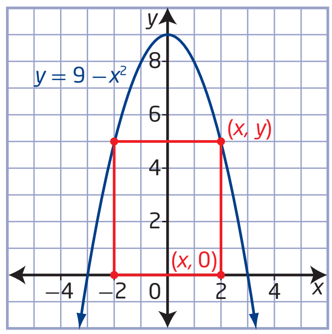
a) $y = -f(-x)$

b) $y = 2f\left(\frac{1}{2}x\right)$

c) $y + 3 = f(x - 4)$

d) $y + 3 = \frac{1}{2}f\left(\frac{1}{4}(x - 4)\right)$

16. A rectangle is inscribed between the x-axis and the parabola $y = 9 - x^2$ with one side along the x-axis, as shown.



a) Write the equation for the area of the rectangle as a function of x.

b) Suppose a horizontal stretch by a factor of 4 is applied to the parabola. What is the equation for the area of the transformed rectangle?

c) Suppose the point (2, 5) is the vertex of the rectangle on the original parabola. Use this point to verify your equations from parts a) and b).

17. The graph of the function $y = 2x^2 + x + 1$ is stretched vertically about the x-axis by a factor of 2, stretched horizontally about the y-axis by a factor of $\frac{1}{3}$, and translated 2 units to the right and 4 units down. Write the equation of the transformed function.

18. This section deals with transformations in a specific order. Give one or more examples of transformations in which the order does not matter. Show how you know that order does not matter.

Create Connections

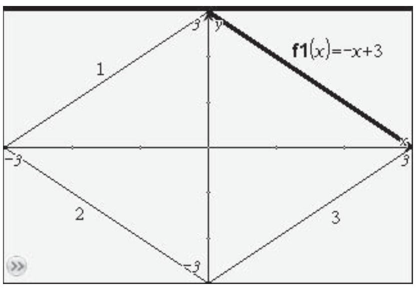
C1 **MINI LAB** Many designs, such as this Moroccan carpet, are based on transformations.

Materials

- grid paper
- graphing calculator

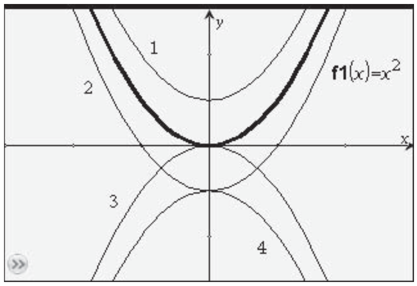
Work with a partner. Use transformations of functions to create designs on a graphing calculator.

Step 1 The graph shows the function $f(x) = -x + 3$ and transformations 1, 2, and 3.



- Recreate the diagram on a graphing calculator. Use the window settings x: [−3, 3, 1] y: [−3, 3, 1].
- Describe the transformations necessary to create the image.
- Write the equations necessary to transform the original function.

Step 2 The graph shows the function $f(x) = x^2$ and transformations 1, 2, 3, and 4.



- Recreate the diagram on a graphing calculator. Use the window settings x: [−3, 3, 1] y: [−3, 3, 1].
- Describe the transformations necessary to create the image.
- Write the equations necessary to transform the original function.

C2 Kokitusi`aki (Diana Passmore) and Siksmissi (Kathy Anderson) make and sell beaded bracelets such as the one shown representing the bear and the wolf.

If they make b bracelets per week at a cost of $f(b)$, what do the following expressions represent? How do they relate to transformations?

a) $f(b + 12)$ **b)** $f(b) + 12$

c) $3f(b)$ **d)** $f(2b)$

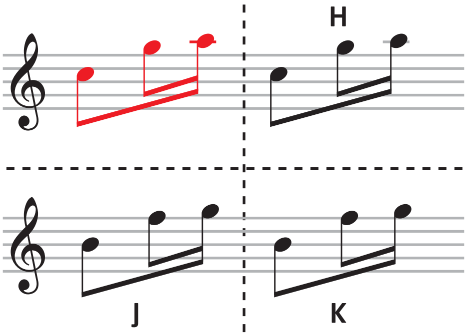
Did You Know?

Sisters Diana Passmore and Kathy Anderson are descendants of the Little Dog Clan of the Piegan (Pikuni'l') Nation of the Blackfoot Confederacy.

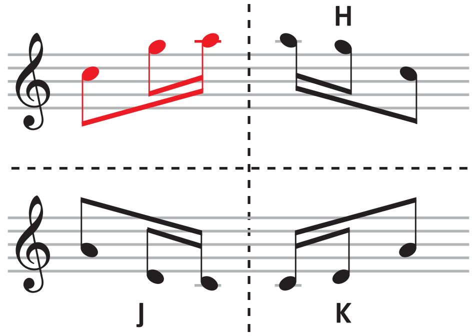
C3 Express the function $y = 2x^2 - 12x + 19$ in the form $y = a(x - h)^2 + k$. Use that form to describe how the graph of $y = x^2$ can be transformed to the graph of $y = 2x^2 - 12x + 19$.

C4 Musical notes can be repeated (translated horizontally), transposed (translated vertically), inverted (horizontal mirror), in retrograde (vertical mirror), or in retrograde inversion (180° rotation). If the musical pattern being transformed is the pattern in red, describe a possible transformation to arrive at the patterns H, J, and K.

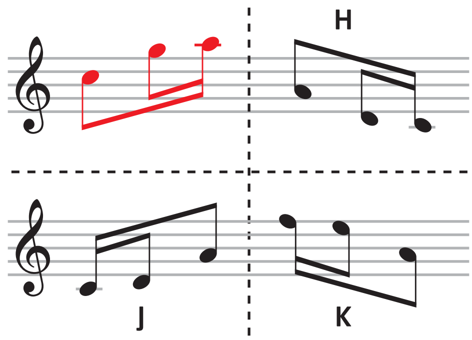
a)



b)

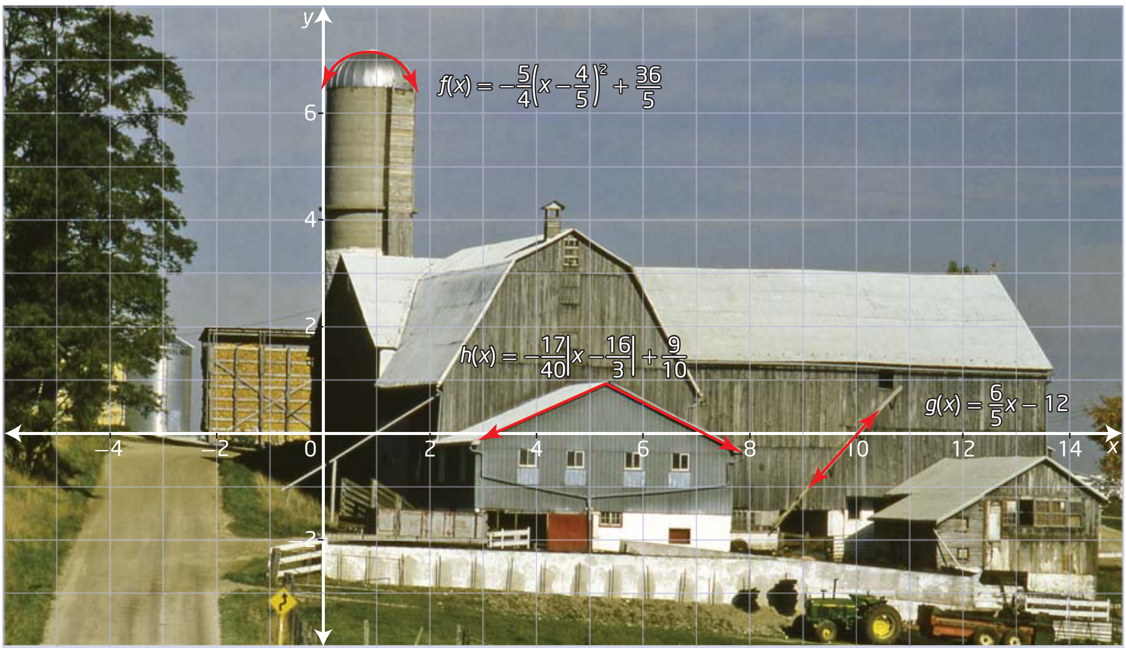


c)



Project Corner — Transformations Around You

- What type(s) of function(s) do you see in the image?
- Describe how each base function has been transformed.



1.4

Inverse of a Relation

Focus on...

- sketching the graph of the inverse of a relation
- determining if a relation and its inverse are functions
- determining the equation of an inverse

An inverse is often thought of as "undoing" or "reversing" a position, order, or effect. Whenever you undo something that you or someone else did, you are using an inverse, whether it is unwrapping a gift that someone else wrapped or closing a door that has just been opened, or deciphering a secret code.

For example, when sending a secret message, a key is used to encode the information. Then, the receiver uses the key to decode the information.

Let each letter in the alphabet be mapped to the numbers 0 to 25.

Plain Text	I	N	V	E	R	S	E
Numeric Values, x	8	13	21	4	17	18	4
Cipher, $x - 2$	6	11	19	2	15	16	2
Cipher Text	G	L	T	C	P	Q	C

Decrypting is the inverse of encrypting. What decryption function would you use on GLTCPQC? What other examples of inverses can you think of?

Investigate the Inverse of a Function

Materials

- grid paper

1. Consider the function $f(x) = \frac{1}{4}x - 5$.

a) Copy the table. In the first column, enter the ordered pairs of five points on the graph of $f(x)$. To complete the second column of the table, interchange the x-coordinates and y-coordinates of the points in the first column.

Key Points on the Graph of $f(x)$	Image Points on the Graph of $g(x)$

b) Plot the points for the function $f(x)$ and draw a line through them.

c) Plot the points for the relation $g(x)$ on the same set of axes and draw a line through them.

2. a) Draw the graph of $y = x$ on the same set of axes as in step 1.

b) How do the distances from the line $y = x$ for key points and corresponding image points compare?

c) What type of transformation occurs in order for $f(x)$ to become $g(x)$?

3. a) What observation can you make about the relationship of the coordinates of your ordered pairs between the graphs of $f(x)$ and g(x)?

b) Determine the equation of $g(x)$. How is this equation related to $f(x) = \frac{1}{4}x - 5$?

c) The relation $g(x)$ is considered to be the inverse of $f(x)$. Is the inverse of $f(x)$ a function? Explain.

Reflect and Respond

4. Describe a way to draw the graph of the **inverse of a function** using reflections.

5. Do you think all inverses of functions are functions? What factors did you base your decision on?

6. a) State a hypothesis for writing the equation of the inverse of a linear function.

b) Test your hypothesis. Write the equation of the inverse of $y = 3x + 2$. Check by graphing.

7. Determine the equation of the inverse of $y = mx + b, m \neq 0$.

a) Make a conjecture about the relationship between the slope of the inverse function and the slope of the original function.

b) Make a conjecture about the relationship between the x-intercepts and the y-intercept of the original function and those of the inverse function.

8. Describe how you could determine if two relations are inverses of each other.

inverse of a function

- if f is a function with domain A and range B, the inverse function, if it exists, is denoted by f^{-1} and has domain B and range A
- f^{-1} maps y to x if and only if f maps x to y

Link the Ideas

The inverse of a relation is found by interchanging the x-coordinates and y-coordinates of the ordered pairs of the relation. In other words, for every ordered pair (x, y) of a relation, there is an ordered pair (y, x) on the inverse of the relation. This means that the graphs of a relation and its inverse are reflections of each other in the line $y = x$.

$(x, y) \rightarrow (y, x)$

Did You Know?

The –1 in $f^{-1}(x)$ does not represent an exponent; that is $f^{-1}(x) \neq \frac{1}{f(x)}$.

The inverse of a function $y = f(x)$ may be written in the form $x = f(y)$. The inverse of a function is not necessarily a function. When the inverse of f is itself a function, it is denoted as f^{-1} and read as "f inverse." When the inverse of a function is not a function, it may be possible to restrict the domain to obtain an inverse function for a portion of the original function.

The inverse of a function reverses the processes represented by that function. Functions $f(x)$ and $g(x)$ are inverses of each other if the operations of $f(x)$ reverse all the operations of $g(x)$ in the opposite order and the operations of $g(x)$ reverse all the operations of $f(x)$ in the opposite order.

For example, $f(x) = 2x + 1$ multiplies the input value by 2 and then adds 1. The inverse function subtracts 1 from the input value and then divides by 2. The inverse function is $f^{-1}(x) = \frac{x - 1}{2}$.

Example 1

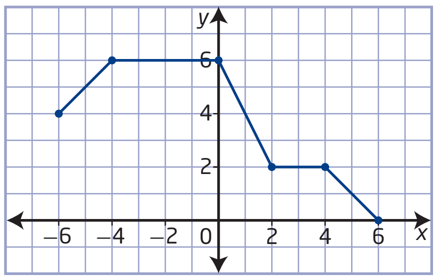
Graph an Inverse

Consider the graph of the relation shown.

a) Sketch the graph of the inverse relation.

b) State the domain and range of the relation and its inverse.

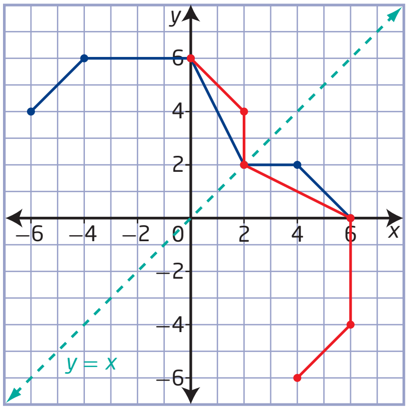
c) Determine whether the relation and its inverse are functions.



Solution

a) To graph the inverse relation, interchange the x-coordinates and y-coordinates of key points on the graph of the relation.

Points on the Relation	Points on the Inverse Relation
(–6, 4)	(4, –6)
(–4, 6)	(6, –4)
(0, 6)	(6, 0)
(2, 2)	(2, 2)
(4, 2)	(2, 4)
(6, 0)	(0, 6)



The graphs are reflections of each other in the line $y = x$. The points on the graph of the relation are related to the points on the graph of the inverse relation by the mapping $(x, y) \rightarrow (y, x)$.

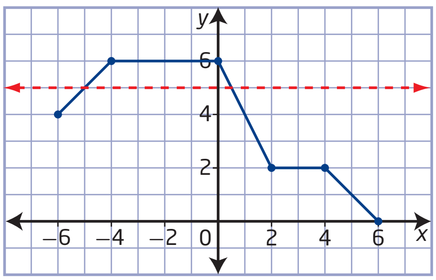
What points are invariant after a reflection in the line $y = x$?

b) The domain of the relation becomes the range of the inverse relation and the range of the relation becomes the domain of the inverse relation.

	Domain	Range
Relation	$\{x \mid -6 \leq x \leq 6, x \in R\}$	$\{y \mid 0 \leq y \leq 6, y \in R\}$
Inverse Relation	$\{x \mid 0 \leq x \leq 6, x \in R\}$	$\{y \mid -6 \leq y \leq 6, y \in R\}$

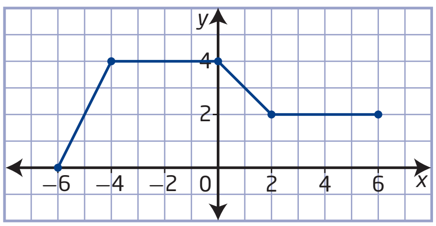
c) The relation is a function of x because there is only one value of y in the range for each value of x in the domain. In other words, the graph of the relation passes the vertical line test.

The inverse relation is not a function of x because it fails the vertical line test. There is more than one value of y in the range for at least one value of x in the domain. You can confirm this by using the **horizontal line test** on the graph of the original relation.



Your Turn

Consider the graph of the relation shown.



a) Determine whether the relation and its inverse are functions.
b) Sketch the graph of the inverse relation.
c) State the domain, range, and intercepts for the relation and the inverse relation.
d) State any invariant points.

Did You Know?

A *one-to-one function* is a function for which every element in the range corresponds to exactly one element in the domain. The graph of a relation is a function if it passes the vertical line test. If, in addition, it passes the horizontal line test, it is a one-to-one function.

horizontal line test

- a test used to determine if the graph of an inverse relation will be a function
- if it is possible for a horizontal line to intersect the graph of a relation more than once, then the inverse of the relation is not a function

Example 2

Restrict the Domain

Consider the function $f(x) = x^2 - 2$.

a) Graph the function $f(x)$. Is the inverse of $f(x)$ a function?

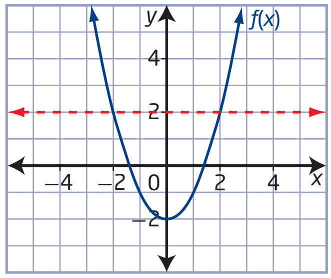
b) Graph the inverse of $f(x)$ on the same set of coordinate axes.

c) Describe how the domain of $f(x)$ could be restricted so that the inverse of $f(x)$ is a function.

Solution

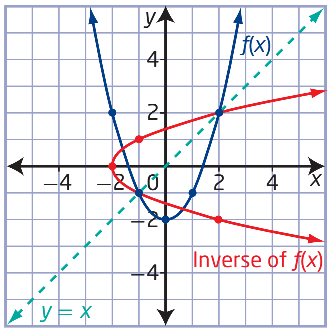
a) The graph of $f(x) = x^2 - 2$ is a translation of the graph of $y = x^2$ by 2 units down.

Since the graph of the function fails the horizontal line test, the inverse of $f(x)$ is not a function.



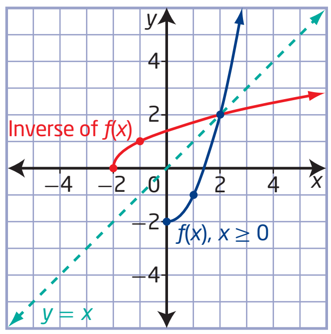
b) Use key points on the graph of $f(x)$ to help you sketch the graph of the inverse of $f(x)$.

Notice that the graph of the inverse of $f(x)$ does not pass the vertical line test. The inverse of $f(x)$ is not a function.



c) The inverse of $f(x)$ is a function if the graph of $f(x)$ passes the horizontal line test.

One possibility is to restrict the domain of $f(x)$ so that the resulting graph is only one half of the parabola. Since the equation of the axis of symmetry is $x = 0$, restrict the domain to $\{x \mid x \geq 0, x \in \mathrm{R}\}$.



How else could the domain of $f(x)$ be restricted?

Your Turn

Consider the function $f(x) = (x + 2)^2$.

a) Graph the function $f(x)$. Is the inverse of $f(x)$ a function?

b) Graph the inverse of $f(x)$ on the same set of coordinate axes.

c) Describe how the domain of $f(x)$ could be restricted so that the inverse of $f(x)$ is a function.

Example 3

Determine the Equation of the Inverse

Algebraically determine the equation of the inverse of each function. Verify graphically that the relations are inverses of each other.

a) $f(x) = 3x + 6$

b) $f(x) = x^2 - 4$

Solution

a) Let $y = f(x)$. To find the equation of the inverse, $x = f(y)$, interchange x and y, and then solve for y.

$f(x) = 3x + 6$

$y = 3x + 6$ — Replace $f(x)$ with y.

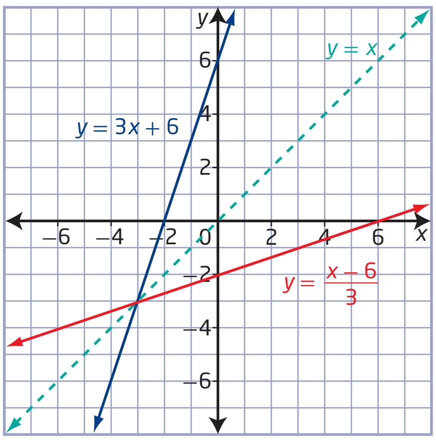
$x = 3y + 6$ — Interchange x and y to determine the inverse.

$x - 6 = 3y$ — Solve for y.

$\frac{x - 6}{3} = y$

$f^{-1}(x) = \frac{x - 6}{3}$ — Replace y with $f^{-1}(x)$, since the inverse of a linear function is also a function.

Graph $y = 3x + 6$ and $y = \frac{x - 6}{3}$ on the same set of coordinate axes.



Notice that the x-intercept and y-intercept of $y = 3x + 6$ become the y-intercept and x-intercept, respectively, of $y = \frac{x - 6}{3}$. Since the functions are reflections of each other in the line $y = x$, the functions are inverses of each other.

b) The same method applies to quadratic functions.

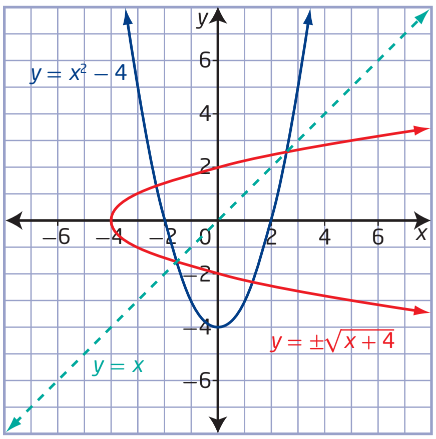
$$f(x) = x^2 - 4$$
$$y = x^2 - 4$$ Replace $f(x)$ with y.
$$x = y^2 - 4$$ Interchange x and y to determine the inverse.
$$x + 4 = y^2$$ Solve for y.
$$\pm\sqrt{x + 4} = y$$
$$y = \pm\sqrt{x + 4}$$

Why is this y not replaced with $f^{-1}(x)$? What could be done so that $f^{-1}(x)$ could be used?

Graph $y = x^2 - 4$ and $y = \pm\sqrt{x + 4}$ on the same set of coordinate axes.

x	$y = x^2 - 4$
−3	5
−2	0
−1	−3
0	−4
1	−3
2	0
3	5

x	$y = \pm\sqrt{x + 4}$
5	±3
0	±2
−3	±1
−4	0



How could you use the tables of values to verify that the relations are inverses of each other?

Notice that the x-intercepts and y-intercept of $y = x^2 - 4$ become the y-intercepts and x-intercept, respectively, of $y = \pm\sqrt{x + 4}$. The relations are reflections of each other in the line $y = x$. While the relations are inverses of each other, $y = \pm\sqrt{x + 4}$ is not a function.

Your Turn

Write the equation for the inverse of the function $f(x) = \frac{x + 8}{3}$. Verify your answer graphically.

Key Ideas

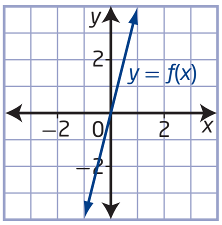
- You can find the inverse of a relation by interchanging the x-coordinates and y-coordinates of the graph.
- The graph of the inverse of a relation is the graph of the relation reflected in the line $y = x$.
- The domain and range of a relation become the range and domain, respectively, of the inverse of the relation.
- Use the horizontal line test to determine if an inverse will be a function.
- You can create an inverse that is a function over a specified interval by restricting the domain of a function.
- When the inverse of a function $f(x)$ is itself a function, it is denoted by $f^{-1}(x)$.
- You can verify graphically whether two functions are inverses of each other.

Check Your Understanding

Practise

1. Copy each graph. Use the reflection line $y = x$ to sketch the graph of $x = f(y)$ on the same set of axes.

a)



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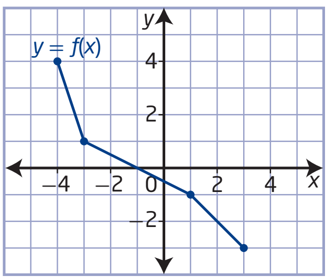
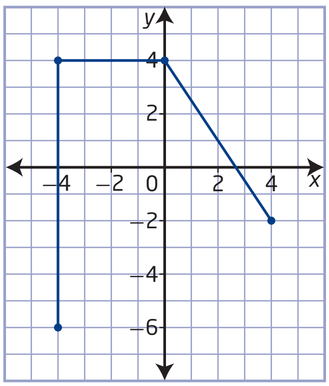
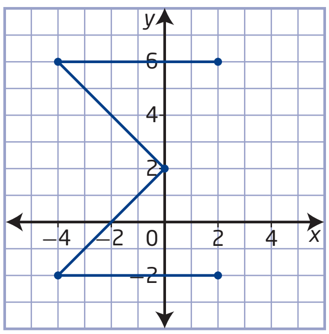


2. Copy the graph of each relation and sketch the graph of its inverse relation.

a)

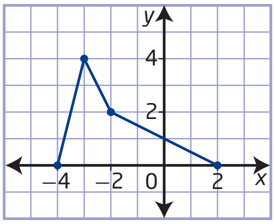


b)



3. State whether or not the graph of the relation is a function. Then, use the horizontal line test to determine whether the inverse relation will be a function.

a)



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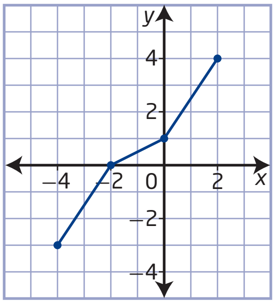
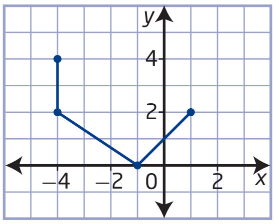
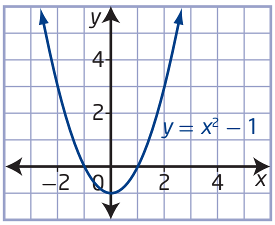


c)

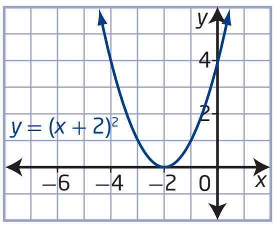


4. For each graph, identify a restricted domain for which the function has an inverse that is also a function.

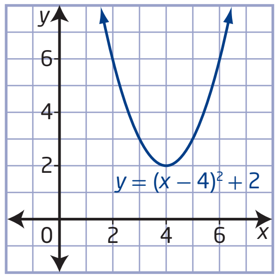
a)



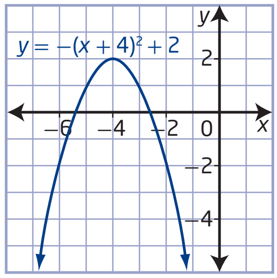
b)



c)



d)



5. Algebraically determine the equation of the inverse of each function.

a) $f(x) = 7x$

b) $f(x) = -3x + 4$

c) $f(x) = \dfrac{x + 4}{3}$

d) $f(x) = \dfrac{x}{3} - 5$

e) $f(x) = 5 - 2x$

f) $f(x) = \dfrac{1}{2}(x + 6)$

6. Match the function with its inverse.

Function

a) $y = 2x + 5$

b) $y = \dfrac{1}{2}x - 4$

c) $y = 6 - 3x$

d) $y = x^2 - 12,\ x \geq 0$

e) $y = \dfrac{1}{2}(x + 1)^2,\ x \leq -1$

Inverse

A $y = \sqrt{x + 12}$

B $y = \dfrac{6 - x}{3}$

C $y = 2x + 8$

D $y = -\sqrt{2x} - 1$

E $y = \dfrac{x - 5}{2}$

Apply

7. For each table, plot the ordered pairs (x, y) and the ordered pairs (y, x). State the domain of the function and its inverse.

a)

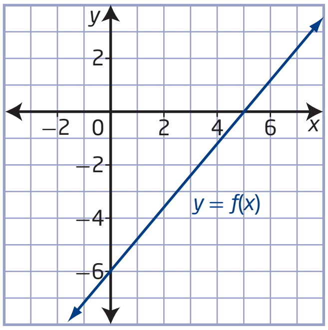
x	y
−2	−2
−1	1
0	4
1	7
2	10

b)

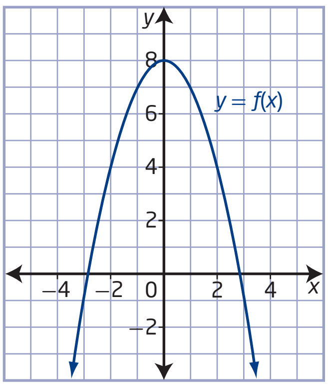
x	y
−6	2
−4	4
−1	5
2	5
5	3

8. Copy each graph of $y = f(x)$ and then sketch the graph of its inverse. Determine if the inverse is a function. Give a reason for your answer.

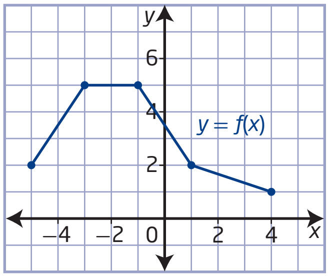
a)



b)



c)



9. For each of the following functions,
- determine the equation for the inverse, $f^{-1}(x)$
- graph $f(x)$ and $f^{-1}(x)$
- determine the domain and range of $f(x)$ and $f^{-1}(x)$

a) $f(x) = 3x + 2$

b) $f(x) = 4 - 2x$

c) $f(x) = \frac{1}{2}x - 6$

d) $f(x) = x^2 + 2,\ x \le 0$

e) $f(x) = 2 - x^2,\ x \ge 0$

10. For each function $f(x)$,

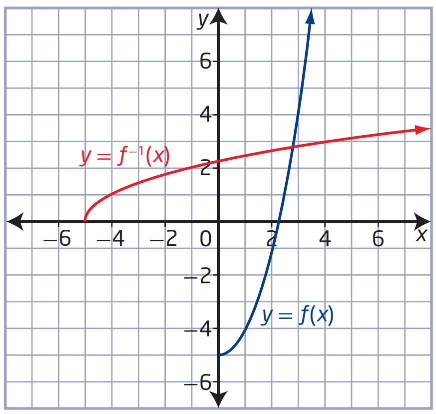
i) determine the equation of the inverse of $f(x)$ by first rewriting the function in the form $y = a(x - h)^2 + k$

ii) graph $f(x)$ and the inverse of $f(x)$

a) $f(x) = x^2 + 8x + 12$

b) $f(x) = x^2 - 4x + 2$

11. Jocelyn and Gerry determine that the inverse of the function $f(x) = x^2 - 5,\ x \ge 0$, is $f^{-1}(x) = \sqrt{x + 5}$. Does the graph verify that these functions are inverses of each other? Explain why.



12. For each of the following functions,

- determine the equation of the inverse
- graph $f(x)$ and the inverse of $f(x)$
- restrict the domain of $f(x)$ so that the inverse of $f(x)$ is a function
- with the domain of $f(x)$ restricted, sketch the graphs of $f(x)$ and $f^{-1}(x)$

a) $f(x) = x^2 + 3$

b) $f(x) = \frac{1}{2}x^2$

c) $f(x) = -2x^2$

d) $f(x) = (x + 1)^2$

e) $f(x) = -(x - 3)^2$

f) $f(x) = (x - 1)^2 - 2$

13. Determine graphically whether the functions in each pair are inverses of each other.

a) $f(x) = x - 4$ and $g(x) = x + 4$

b) $f(x) = 3x + 5$ and $g(x) = \frac{x - 5}{3}$

c) $f(x) = x - 7$ and $g(x) = 7 - x$

d) $f(x) = \frac{x - 2}{2}$ and $g(x) = 2x + 2$

e) $f(x) = \frac{8}{x - 7}$ and $g(x) = \frac{8}{x + 7}$

14. For each function, state two ways to restrict the domain so that the inverse is a function.

a) $f(x) = x^2 + 4$

b) $f(x) = 2 - x^2$

c) $f(x) = (x - 3)^2$

d) $f(x) = (x + 2)^2 - 4$

15. Given the function $f(x) = 4x - 2$, determine each of the following.

a) $f^{-1}(4)$

b) $f^{-1}(-2)$

c) $f^{-1}(8)$

d) $f^{-1}(0)$

16. The function for converting the temperature from degrees Fahrenheit, x, to degrees Celsius, y, is $y = \frac{5}{9}(x - 32)$.

a) Determine the equivalent temperature in degrees Celsius for 90 °F.

b) Determine the inverse of this function. What does it represent? What do the variables represent?

c) Determine the equivalent temperature in degrees Fahrenheit for 32 °C.

d) Graph both functions. What does the invariant point represent in this situation?

17. A forensic specialist can estimate the height of a person from the lengths of their bones. One function relates the length, x, of the femur to the height, y, of the person, both in centimetres.

For a male: $y = 2.32x + 65.53$
For a female: $y = 2.47x + 54.13$

a) Determine the height of a male and of a female with a femur length of 45.47 cm.

b) Use inverse functions to determine the femur length of

i) a male whose height is 187.9 cm

ii) a female whose height is 175.26 cm

18. In Canada, ring sizes are specified using a numerical scale. The numerical ring size, y, is approximately related to finger circumference, x, in millimetres, by $y = \frac{x - 36.5}{2.55}$.

a) What whole-number ring size corresponds to a finger circumference of 49.3 mm?

b) Determine an equation for the inverse of the function. What do the variables represent?

c) What finger circumferences correspond to ring sizes of 6, 7, and 9?

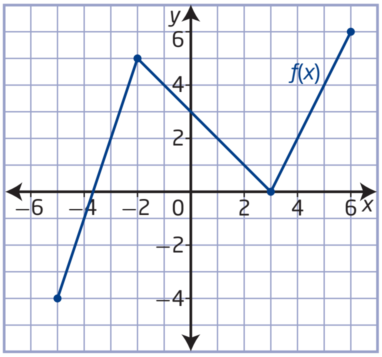
Extend

19. When a function is constantly increasing or decreasing, its inverse is a function. For each graph of $f(x)$,

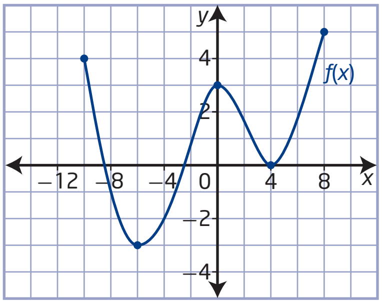
i) choose an interval over which the function is increasing and sketch the inverse of the function when it is restricted to that domain

ii) choose an interval over which the function is decreasing and sketch the inverse of the function when it is restricted to that domain

a)



b)



20. Suppose a function $f(x)$ has an inverse function, $f^{-1}(x)$.

a) Determine $f^{-1}(5)$ if $f(17) = 5$.

b) Determine $f(-2)$ if $f^{-1}(\sqrt{3}) = -2$.

c) Determine the value of a if $f^{-1}(a) = 1$ and $f(x) = 2x^2 + 5x + 3$, $x \geq -1.25$.

21. If the point (10, 8) is on the graph of the function $y = f(x)$, what point must be on the graph of each of the following?

a) $y = f^{-1}(x + 2)$

b) $y = 2f^{-1}(x) + 3$

c) $y = -f^{-1}(-x) + 1$

Create Connections

C1 Describe the inverse sequence of operations for each of the following.

a) $f(x) = 6x + 12$

b) $f(x) = (x + 3)^2 - 1$

C2 **a)** Sketch the graphs of the function $f(x) = -x + 3$ and its inverse, $f^{-1}(x)$.

b) Explain why $f(x) = f^{-1}(x)$.

c) If a function and its inverse are the same, how are they related to the line $y = x$?

C3 Two students are arguing about whether or not a given relation and its inverse are functions. Explain how the students could verify who is correct.

C4 **MINI LAB** Two functions, $f(x) = \frac{x + 5}{3}$ and $g(x) = 3x - 5$, are inverses of each other.

Step 1 Evaluate output values for $f(x)$ for $x = 1$, $x = 4$, $x = -8$, and $x = a$. Use the results as input values for $g(x)$. What do you notice about the output values for $g(x)$? Explain why this happens. State a hypothesis that could be used to verify whether or not two functions are inverses of each other.

Step 2 Reverse the order in which you used the functions. Start with using the input values for $g(x)$, and then use the outputs in $f(x)$. What conclusion can you make about inverse functions?

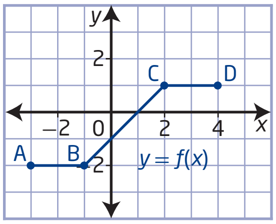
Step 3 Test your conclusions and hypothesis by selecting two functions of your own.

Step 4 Explain how your results relate to the statement "if $f(a) = b$ and $f^{-1}(b) = a$, then the two functions are inverses of each other." Note that this must also be true when the function roles are switched.

Chapter 1 Review

1.1 Horizontal and Vertical Translations, pages 6–15

1. Given the graph of the function $y = f(x)$, sketch the graph of each transformed function.

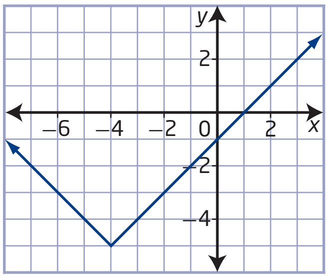


a) $y - 3 = f(x)$

b) $h(x) = f(x + 1)$

c) $y + 1 = f(x - 2)$

2. Describe how to translate the graph of $y = |x|$ to obtain the graph of the function shown. Write the equation of the transformed function in the form $y - k = |x - h|$.



3. The range of the function $y = f(x)$ is $\{y \mid -2 \leq y \leq 5, y \in \mathrm{R}\}$. What is the range of the function $y = f(x - 2) + 4$?

4. James wants to explain vertical and horizontal translations by describing the effect of the translation on the coordinates of a point on the graph of a function. He says, "If the point (a, b) is on the graph of $y = f(x)$, then the point $(a - 5, b + 4)$ is the image point on the graph of $y + 4 = f(x - 5)$." Do you agree with James? Explain your reasoning.

1.2 Reflections and Stretches, pages 16–31

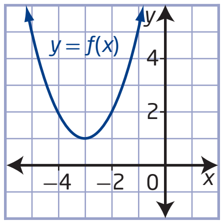
5. Name the line of reflection when the graph of $y = f(x)$ is transformed as indicated. Then, state the coordinates of the image point of (3, 5) on the graph of each reflection.

a) $y = -f(x)$

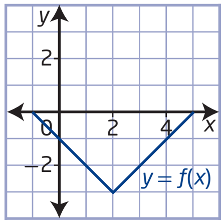
b) $y = f(-x)$

6. Copy each graph of $y = f(x)$. Then,
- sketch the reflection indicated
- state the domain and range of the transformed function
- list any invariant points

a) $y = f(-x)$



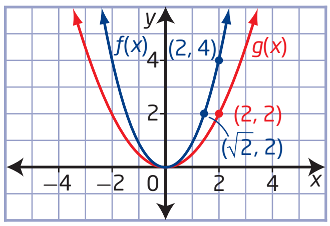
b) $y = -f(x)$



7. a) Sketch the graphs of the functions $f(x) = x^2$, $g(x) = f(2x)$, and $h(x) = f\left(\frac{1}{2}x\right)$ on the same set of coordinate axes.

b) Describe how the value of the coefficient of x for $g(x)$ and $h(x)$ affects the graph of the function $f(x) = x^2$.

8. Consider the graphs of the functions $f(x)$ and $g(x)$.

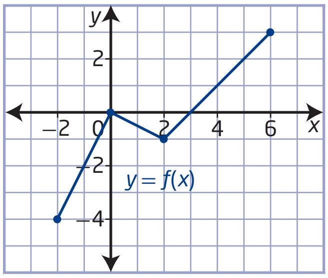


a) Is the graph of $g(x)$ a horizontal or a vertical stretch of the graph of $f(x)$? Explain your reasoning.

b) Write the equation that models the graph of $g(x)$ as a transformation of the graph of $f(x)$.

1.3 Combining Transformations, pages 32–43

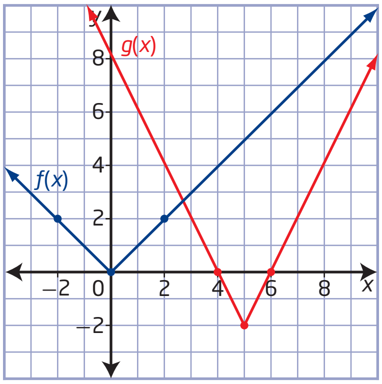
9. Given the graph of $y = f(x)$, sketch the graph of each transformed function.



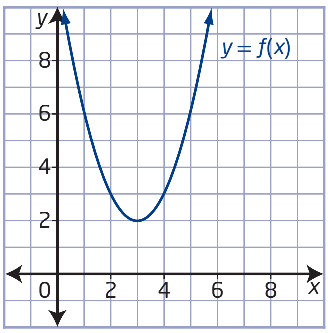
a) $y = 2f\left(\frac{1}{2}x\right)$ **b)** $y = \frac{1}{2}f(3x)$

10. Explain how the transformations described by $y = f(4(x + 1))$ and $y = f(4x + 1)$ are similar and how they are different.

11. Write the equation for the graph of $g(x)$ as a transformation of the equation for the graph of $f(x)$.



12. Consider the graph of $y = f(x)$. Sketch the graph of each transformation.



a) $y = \frac{1}{2}f(-(x + 2))$

b) $y - 2 = -f(2(x - 3))$

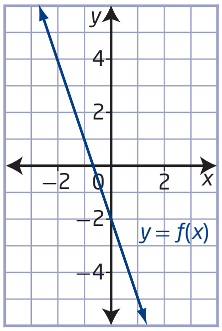
c) $y - 1 = 3f(2x + 4)$

1.4 Inverse of a Relation, pages 44–55

13. a) Copy the graph of $y = f(x)$ and sketch the graph of $x = f(y)$.

b) Name the line of reflection and list any invariant points.

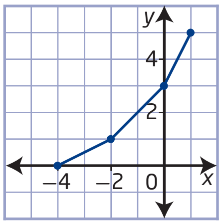
c) State the domain and range of the two functions.

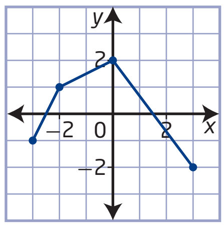


14. Copy and complete the table.

$y = f(x)$		$y = f^{-1}(x)$	
x	**y**	**x**	**y**
−3	7		
		4	2
10		−12	

15. Sketch the graph of the inverse relation for each graph. State whether the relation and its inverse are functions.

a)


b)


16. Algebraically determine the equation of the inverse of the function $y = (x - 3)^2 + 1$. Determine a restriction on the domain of the function in order for its inverse to be a function. Show your thinking.

17. Graphically determine if the functions are inverses of each other.

a) $f(x) = -6x + 5$ and $g(x) = \frac{x + 5}{6}$

b) $f(x) = \frac{x - 3}{8}$ and $g(x) = 8x + 3$

Chapter 1 Practice Test

Multiple Choice

For #1 to #7, choose the best answer.

1. What is the effect on the graph of the function $y = x^2$ when the equation is changed to $y = (x + 1)^2$?

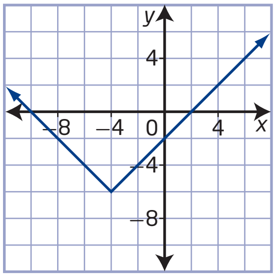
A The graph is stretched vertically.

B The graph is stretched horizontally.

C The graph is the same shape but translated up.

D The graph is the same shape but translated to the left.

2. The graph shows a transformation of the graph of $y = |x|$. Which equation models the graph?



A $y + 4 = |x - 6|$

B $y - 6 = |x - 4|$

C $y - 4 = |x + 6|$

D $y + 6 = |x + 4|$

3. If (a, b) is a point on the graph of $y = f(x)$, which of the following points is on the graph of $y = f(x + 2)$?

A $(a + 2, b)$

B $(a - 2, b)$

C $(a, b + 2)$

D $(a, b - 2)$

4. Which equation represents the image of $y = x^2 + 2$ after a reflection in the y-axis?

A $y = -x^2 - 2$

B $y = x^2 + 2$

C $y = -x^2 + 2$

D $y = x^2 - 2$

5. The effect on the graph of $y = f(x)$ if it is transformed to $y = \frac{1}{4}f(3x)$ is

A a vertical stretch by a factor of $\frac{1}{4}$ and a horizontal stretch by a factor of 3

B a vertical stretch by a factor of $\frac{1}{4}$ and a horizontal stretch by a factor of $\frac{1}{3}$

C a vertical stretch by a factor of 4 and a horizontal stretch by a factor of 3

D a vertical stretch by a factor of 4 and a horizontal stretch by a factor of $\frac{1}{3}$

6. Which of the following transformations of $f(x)$ produces a graph that has the same y-intercept as $f(x)$? Assume that $(0, 0)$ is not a point on $f(x)$.

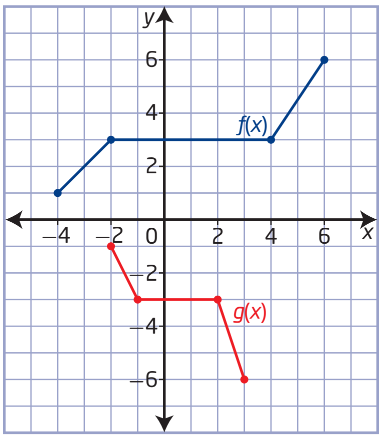
A $-9f(x)$

B $f(x) - 9$

C $f(-9x)$

D $f(x - 9)$

7. Given the graphs of $y = f(x)$ and $y = g(x)$, what is the equation for $g(x)$ in terms of $f(x)$?



A $g(x) = f\left(-\frac{1}{2}x\right)$

B $g(x) = f(-2x)$

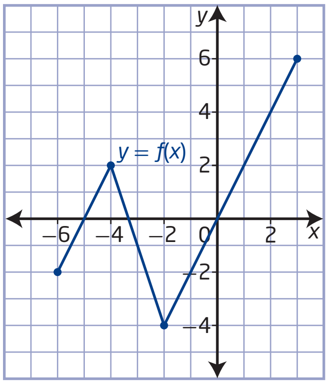
C $g(x) = -f(2x)$

D $g(x) = -f\left(\frac{1}{2}x\right)$

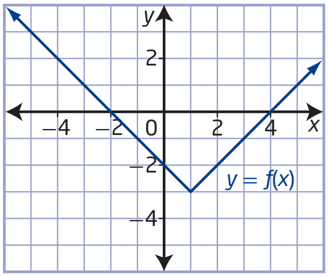
Short Answer

8. The domain of the function $y = f(x)$ is $\{x \mid -3 \le x \le 4, x \in R\}$. What is the domain of the function $y = f(x + 2) - 1$?

9. Given the graph of $y = f(x)$, sketch the graph of $y - 4 = -\frac{1}{4}f\left(\frac{1}{2}(x + 3)\right)$.



10. Consider the graph of the function $y = f(x)$.



a) Sketch the graph of the inverse.

b) Explain how the coordinates of key points are transformed.

c) State any invariant points.

11. Write the equation of the inverse function of $y = 5x + 2$. Verify graphically that the functions are inverses of each other.

12. A transformation of the graph of $y = f(x)$ results in a horizontal stretch about the y-axis by a factor of 2, a horizontal reflection in the y-axis, a vertical stretch about the x-axis by a factor of 3, and a horizontal translation of 2 units to the right. Write the equation for the transformed function.

Extended Response

13. The graph of the function $f(x) = |x|$ is transformed to the graph of $g(x) = f(x + 2) - 7$.

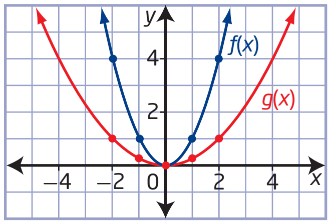
a) Describe the transformation.

b) Write the equation of the function $g(x)$.

c) Determine the minimum value of $g(x)$.

d) The domain of the function $f(x)$ is the set of real numbers. The domain of the function $g(x)$ is also the set of real numbers. Does this imply that all of the points are invariant? Explain your answer.

14. The function $g(x)$ is a transformation of the function $f(x)$.



a) Write the equation of the function $f(x)$.

b) Write the equation of the function $g(x)$ in the form $g(x) = af(x)$, and describe the transformation.

c) Write the equation of the function $g(x)$ in the form $g(x) = f(bx)$, and describe the transformation.

d) Algebraically prove that the two equations from parts b) and c) are equivalent.

15. Consider the function $h(x) = -(x + 3)^2 - 5$.

a) Explain how you can determine whether or not the inverse of $h(x)$ is a function.

b) Write the equation of the inverse relation in simplified form.

c) What restrictions could be placed on the domain of the function so that the inverse is also a function?