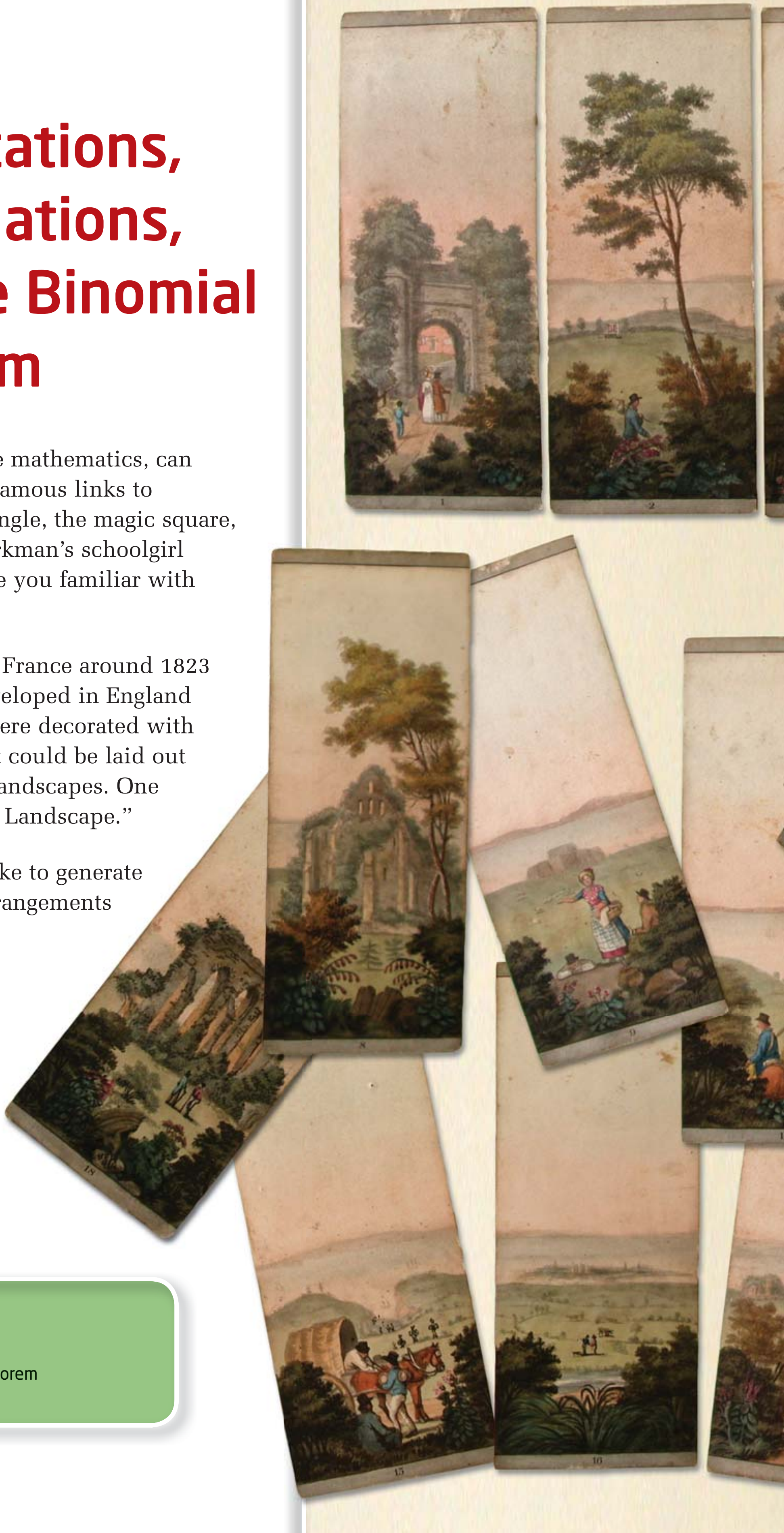
CHAPTER 11

Permutations, Combinations, and the Binomial Theorem

Combinatorics, a branch of discrete mathematics, can be defined as the art of counting. Famous links to combinatorics include Pascal's triangle, the magic square, the Königsberg bridge problem, Kirkman's schoolgirl problem, and myriorama cards. Are you familiar with any of these?

Myriorama cards were invented in France around 1823 by Jean-Pierre Brès and further developed in England by John Clark. Early myrioramas were decorated with people, buildings, and scenery that could be laid out in any order to create a variety of landscapes. One 24-card set is sold as "The Endless Landscape."

How long do you think it would take to generate the 6.2×10^{23} possible different arrangements from a 24-card myriorama set?



Key Terms

fundamental counting principle
factorial
permutation
combination
binomial theorem

Career Link

Actuaries are business professionals who calculate the likelihood of events, especially those involving risk to a business or government. They use their mathematical skills to devise ways of reducing the chance of negative events occurring and lessening their impact should they occur. This information is used by insurance companies to set rates and by corporations to minimize the negative effects of risk-taking. The work is as challenging as correctly predicting the future!

Web Link

To find out more about the career of an actuary, go to www.mcgrawhill.ca/school/learningcentres and follow the links.

11.1

Permutations

Focus on…

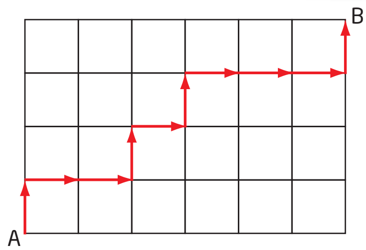
- solving counting problems using the fundamental counting principle
- determining, using a variety of strategies, the number of permutations of *n* elements taken *r* at a time
- solving counting problems when two or more elements are identical
- solving an equation that involves $_nP_r$ notation

How safe is your password? It has been suggested that a four-character letters-only password can be hacked in under 10 s. However, an eight-character password with at least one number could take up to 7 years to crack. Why is there such a big difference?

In how many possible ways can you walk from A to B in a four by six “rectangular city” if you must walk on the grid lines and move only up or to the right?

The diagram shows one successful path from A to B. What strategies might help you solve this problem?

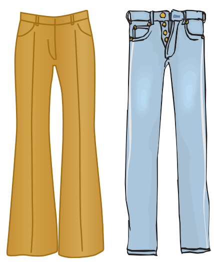
You will learn how to solve problems like these in this section.



Investigate Possible Arrangements

You are packing clothing to go on a trip. You decide to take three different tops and two pairs of pants.

1. If all of the items go together, how many different outfits can you make? Show how to get the answer using different strategies. Discuss your strategies with a partner.



2. You also take two pairs of shoes. How many different outfits consisting of a top, a pair of pants, and a pair of shoes are possible?
3. **a)** Determine the number of different outfits you can make when you take four pairs of pants, two shirts, and two hats, if an outfit consists of a pair of pants, a shirt, and a hat.

   b) Check your answer using a tree diagram.

Reflect and Respond

4. Make a conjecture about how you can use multiplication only to arrive at the number of different outfits possible in steps 1 to 3.

5. A friend claims he can make 1000 different outfits using only tops, pants, and shoes. Show how your friend could be correct.

Link the Ideas

Counting methods are used to determine the number of members of a specific set as well as the outcomes of an event. You can display all of the possible choices using tables, lists, or tree diagrams and then count the number of outcomes. Another method of determining the number of possible outcomes is to use the **fundamental counting principle**.

fundamental counting principle

- if one task can be performed in *a* ways and a second task can be performed in *b* ways, then the two tasks can be performed in $a \times b$ ways
- for example, a restaurant meal consists of one of two salad options, one of three entrees, and one of four desserts, so there are (2)(3)(4) or 24 possible meals

Example 1

Arrangements With or Without Restrictions

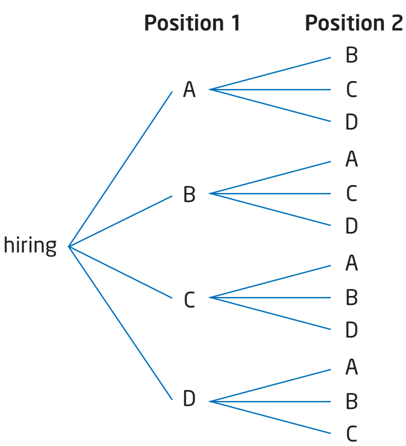
a) A store manager has selected four possible applicants for two different positions at a department store. In how many ways can the manager fill the positions?

b) In how many ways can a teacher seat four girls and three boys in a row of seven seats if a boy must be seated at each end of the row?

Solution

a) Method 1: List Outcomes and Count the Total

Use a tree diagram and count the outcomes, or list all of the hiring choices in a table. Let A represent applicant 1, B represent applicant 2, C represent applicant 3, and D represent applicant 4.



Position 1	Position 2
A	B
A	C
A	D
B	A
B	C
B	D
C	A
C	B
C	D
D	A
D	B
D	C

Total pathways = 12

12 possibilities

There are 12 possible ways to fill the 2 positions.

Method 2: Use the Fundamental Counting Principle

■	■
(number of choices for position 1)	(number of choices for position 2)

If the manager chooses a person for position 1, then there are four choices. Once position 1 is filled, there are only three choices left for position 2.

4	3
(number of choices for position 1)	(number of choices for position 2)

According to the fundamental counting principle, there are (4)(3) or 12 ways to fill the positions.

b) Use seven blanks to represent the seven seats in the row.

■	■	■	■	■	■	■
(Seat 1)	(Seat 2)	(Seat 3)	(Seat 4)	(Seat 5)	(Seat 6)	(Seat 7)

There is a restriction: a boy must be in each end seat. Fill seats 1 and 7 first.

Why do you fill these two seats first?

If the teacher starts with seat 1, there are three boys to choose. Once the teacher fills seat 1, there are two choices for seat 7.

3	■	■	■	■	■	2
(Seat 1)	(Seat 2)	(Seat 3)	(Seat 4)	(Seat 5)	(Seat 6)	(Seat 7)
Boy						Boy

Once the end seats are filled, there are five people (four girls and one boy) to arrange in the seats as shown.

Why do you not need to distinguish between boys and girls for the second through sixth seats?

3	5	4	3	2	1	2
(Seat 1)	(Seat 2)	(Seat 3)	(Seat 4)	(Seat 5)	(Seat 6)	(Seat 7)

By the fundamental counting principle, the teacher can arrange the girls and boys in $(3)(5)(4)(3)(2)(1)(2) = 720$ ways.

Your Turn

Use any method to solve each problem.

a) How many three-digit numbers can you make using the digits 1, 2, 3, 4, and 5? Repetition of digits is not allowed.

b) How does the application of the fundamental counting principle in part a) change if repetition of the digits is allowed? Determine how many three-digit numbers can be formed that include repetitions.

factorial

- for any positive integer n, the product of all of the positive integers up to and including n
- $4! = (4)(3)(2)(1)$
- $0!$ is defined as 1

In Example 1b), the remaining five people (four girls and one boy) can be arranged in (5)(4)(3)(2)(1) ways. This product can be abbreviated as 5! and is read as "five **factorial**."

Therefore, $5! = (5)(4)(3)(2)(1)$.
In general, $n! = (n)(n - 1)(n - 2)\ldots(3)(2)(1)$, where $n \in \mathrm{N}$.

The arrangement of objects or people in a line is called a linear **permutation**. In a permutation, the order of the objects is important. When the objects are distinguishable from one another, a new order of objects creates a new permutation.

Seven different objects can be arranged in 7! ways.

$7! = (7)(6)(5)(4)(3)(2)(1)$

Explain why 7! is equivalent to 7(6!) or to 7(6)(5)(4!).

If there are seven members on the student council, in how many ways can the council select three students to be the chair, the secretary, and the treasurer of the council?

Using the fundamental counting principle, there are (7)(6)(5) possible ways to fill the three positions. Using the factorial notation,

$$\begin{aligned}\frac{7!}{4!} &= \frac{(7)(6)(5)\cancel{(4)}\cancel{(3)}\cancel{(2)}\cancel{(1)}}{\cancel{(4)}\cancel{(3)}\cancel{(2)}\cancel{(1)}}\\ &= (7)(6)(5)\\ &= 210\end{aligned}$$

> The notation ${}_nP_r$ is used to represent the number of permutations, or arrangements in a definite order, of r items taken from a set of n distinct items. A formula for ${}_nP_r$ is ${}_nP_r = \frac{n!}{(n-r)!}$, $n \in N$.

Using permutation notation, ${}_7P_3$ represents the number of arrangements of three objects taken from a set of seven objects.

$$\begin{aligned}{}_7P_3 &= \frac{7!}{(7-3)!}\\ &= \frac{7!}{4!}\\ &= 210\end{aligned}$$

So, there are 210 ways that the 3 positions can be filled from the 7-member council.

Example 2

Using Factorial Notation

a) Evaluate ${}_9P_4$ using factorial notation.
b) Show that $100! + 99! = 101(99!)$ without using technology.
c) Solve for n if ${}_nP_3 = 60$, where n is a natural number.

Solution

a)
$$\begin{aligned}{}_9P_4 &= \frac{9!}{(9-4)!}\\ &= \frac{9!}{5!}\\ &= \frac{(9)(8)(7)(6)\cancel{5!}}{\cancel{5!}}\\ &= (9)(8)(7)(6)\\ &= 3024\end{aligned}$$

Why is 9! the same as (9)(8)(7)(6)5!?

permutation

- an ordered arrangement or sequence of all or part of a set
- for example, the possible permutations of the letters A, B, and C are ABC, ACB, BAC, BCA, CAB, and CBA

Did You Know?

The notation *n*! was introduced in 1808 by Christian Kramp (1760–1826) as a convenience to the printer. Until then, $\underline{n|}$ had been used.

Did You Know?

Most scientific and graphing calculators can evaluate factorials and calculate the number of permutations for *n* distinct objects taken *r* at a time. Learn to use these features on the calculator you use.

b) $100! + 99! = 100(99!) + 99!$
$= 99!(100 + 1)$
$= 99!(101)$
$= 101(99!)$

What math technique was used in going from step 1 to step 2?

c)
$$_nP_3 = 60$$
$$\frac{n!}{(n-3)!} = 60$$
$$\frac{n(n-1)(n-2)(n-3)!}{(n-3)!} = 60$$
$$n(n-1)(n-2) = 60$$

Why must $n \geq 3$?

Method 1: Use Reasoning

$n(n-1)(n-2) = 60$
$n(n-1)(n-2) = 5(4)(3)$
$n = 5$

Why is 60 rewritten as the product of three consecutive natural numbers?
Will any other values work for n? Why or why not?

The solution to $_nP_3 = 60$ is $n = 5$.

Method 2: Use Algebra

$n(n-1)(n-2) = 60$
$n^3 - 3n^2 + 2n - 60 = 0$

Since n must be a natural number, only factors of 60 that are natural numbers must be considered: 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60.

Test these factors using the factor theorem.
$P(n) = n^3 - 3n^2 + 2n - 60$
$P(5) = 5^3 - 3(5)^2 + 2(5) - 60$
$= 0$

Therefore, $n = 5$ is a solution.

Test $n = 5$ in the original equation.

Left Side $= n(n-1)(n-2)$
$= 5(5-1)(5-2)$
$= 60$

Right Side $= 60$

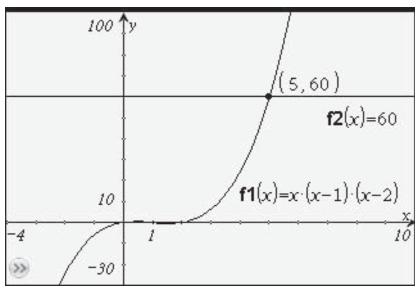
Do you need to test any other values for n? Why or why not?

The solution to $_nP_3 = 60$ is $n = 5$.

Which of the three methods do you prefer?

Method 3: Use Graphing

Graph to solve the equation $n(n-1)(n-2) = 60$.
Graph $y = n(n-1)(n-2)$ and $y = 60$ and find the point of intersection.



Where does the solution for the original equation occur on your graph?

The solution to $_nP_3 = 60$ is $n = 5$.

Your Turn

a) Evaluate $_7P_2$ using factorial notation.
b) Show that $5! - 3! = 19(3!)$.
c) Solve for n if $_nP_2 = 56$.

Permutations With Repeating Objects

Consider the number of four-letter arrangements possible using the letters from the word *pool*.

pool opol oopl oolp polo oplo olpo olop ploo lpoo lopo loop
pool opol oopl oolp polo oplo olpo olop ploo lpoo lopo loop

If all of the letters were different, the number of possible four-letter arrangements would be $4! = 24$.

There are two identical letters (*o*), which, if they were different, could be arranged in $2! = 2$ ways.

The number of four-letter arrangements possible when two of the letters are the same is $\frac{4!}{2!} = \frac{24}{2}$ or 12.

Why do you divide by 2!?

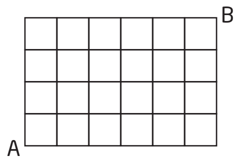
> A set of n objects with a of one kind that are identical, b of a second kind that are identical, and c of a third kind that are identical, and so on, can be arranged in $\frac{n!}{a!b!c!\ldots}$ different ways.

Example 3

Repeating Objects

a) How many different eight-letter arrangements can you make using the letters of *aardvark*?

b) How many paths can you follow from A to B in a four by six rectangular grid if you move only up or to the right?



Solution

a) There are eight letters in *aardvark*. There are 8! ways to arrange eight letters. But of the eight letters, three are the letter *a* and two are the letter *r*. There are 3! ways to arrange the *a*'s and 2! ways to arrange the *r*'s. The number of different eight-letter arrangements is $\frac{8!}{3!2!} = 3360$.

b) Each time you travel 1 unit up, it is the same distance no matter where you are on the grid. Similarly, each horizontal movement is the same distance to the right. So, using U to represent 1 unit up and R to represent 1 unit to the right, one possible path is UUUURRRRRR. The problem is to find the number of arrangements of UUUURRRRRR.

For every path from A to B, how many units of distance must you travel?

How many vertical units must you travel?

How many horizontal units must you travel?

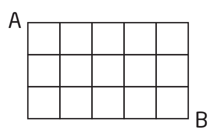
The number of different paths is $\frac{10!}{4!6!} = 210$.

Where did the numbers 10, 4, and 6 come from?

Your Turn

a) How many different 5-digit numbers can you make by arranging all of the digits of 17 171?

b) In how many different ways can you walk from A to B in a three by five rectangular grid if you must move only down or to the right?



Example 4

Permutations with Constraints

Five people (A, B, C, D, and E) are seated on a bench. In how many ways can they be arranged if

a) E is seated in the middle?

b) A and B must be seated together?

c) A and B cannot be together?

Solution

a) Since E must be in the middle, there is only 1 choice for that position. This leaves four people to be arranged in (4)(3)(2)(1) ways.

4	3	1	2	1
(Seat 1)	(Seat 2)	(Seat 3)	(Seat 4)	(Seat 5)
		Middle		

What is the restriction?

There are $(4)(3)(1)(2)(1) = 24$ ways to arrange the five people with E seated in the middle.

b) There are 2! ways to arrange A and B together, AB or BA.
Consider A and B together as 1 object. This means that there are 4 objects (C, D, E, and AB) to arrange in $4! = 24$ ways.
Then, there are $2!4! = 48$ ways to arrange five people if A and B must be seated together.

c) **Method 1: Use Positions When A and B Are Not Together**
There are five positions on the bench. A and B are not together when they are in the following positions:

1st and 3rd 1st and 4th 1st and 5th
2nd and 4th 2nd and 5th 3rd and 5th (6 ways)

For any one of these six arrangements, A and B can be interchanged. (2 ways)

The remaining 3 people can always be arranged 3! or 6 ways. (6 ways)
There are $(6)(2)(6) = 72$ ways where A and B are not seated together.

Why is it necessary to multiply to get the final answer?

Method 2: Use Positions When A and B Are Together
The total number of arrangements for five people in a row with no restrictions is $5! = 120$. Arrangements with A and B together is 48 from part b).
Therefore, the number of arrangements with A and B not together is
Total number of arrangements − Number of arrangements together
$= 5! - 2!4!$
$= 120 - 48$
$= 72$

Your Turn

How many ways can one French poster, two mathematics posters, and three science posters be arranged in a row on a wall if

a) the two mathematics posters must be together on an end?

b) the three science posters must be together?

c) the three science posters cannot all be together?

Arrangements Requiring Cases

To solve some problems, you must count the different arrangements in cases. For example, you might need to determine the number of arrangements of four girls and three boys in a row of seven seats if the ends of the rows must be either both female or both male.

Case 1: Girls on Ends of Rows — **Arrangements**

Girl	(2 Girls and 3 Boys)	Girl	
4	5!	3	$(4)(5!)(3) = 1440$

Case 2: Boys on Ends of Rows

Boy	(4 Girls and 1 Boy)	Boy	
3	5!	2	$(3)(5!)(2) = 720$

Total number of arrangements: $1440 + 720 = 2160$

Example 5

Using Cases to Determine Permutations

How many different 3-digit even numbers greater than 300 can you make using the digits 1, 2, 3, 4, 5, and 6? No digits are repeated.

Solution

When determining the number of permutations for a situation in which there are restrictions, you must first address the choices with the restrictions.

Why does the solution to this example require the identification of cases?

Case 1: Numbers That Are Even and Start With 3 or 5

Numbers start with 3 or 5, so there are two choices for the first digit.

Numbers are even, so there are three choices for the third digit.

Number of choices for first digit	Number of choices for second digit	Number of choices for third digit
2	4	3

How do you know there are four possible choices for the middle digit?

$$\begin{aligned}\text{Number of possibilities} &= 2(4)(3)\\ &= 24\end{aligned}$$

Case 2: Numbers That Are Even and Start With 4 or 6

Numbers start with 4 or 6, so there are two choices for the first digit.

Numbers are even, so two choices remain for the third digit.

Number of choices for first digit	Number of choices for second digit	Number of choices for third digit
2	4	2

Why are there only two choices for the third digit?

$$\begin{aligned}\text{Number of possibilities} &= 2(4)(2)\\ &= 16\end{aligned}$$

The final answer is the sum of the possibilities from the two cases.

There are 24 + 16, or 40, 3-digit even numbers greater than 300.

Your Turn

How many 4-digit odd numbers can you make using the digits 1 to 7 if the numbers must be less than 6000? No digits are repeated.

Key Ideas

- The fundamental counting principle can be used to determine the number of different arrangements. If one task can be performed in a ways, a second task in b ways, and a third task in c ways, then all three tasks can be arranged in $a \times b \times c$ ways.
- Factorial notation is an abbreviation for products of successive positive integers.

  $5! = (5)(4)(3)(2)(1)$
  $(n + 1)! = (n + 1)(n)(n - 1)(n - 2)\cdots(3)(2)(1)$
- A permutation is an arrangement of objects in a definite order. The number of permutations of n different objects taken r at a time is given by ${}_nP_r = \frac{n!}{(n - r)!}$.
- A set of n objects containing a identical objects of one kind, b identical objects of another kind, and so on, can be arranged in $\frac{n!}{a!b!\ldots}$ ways.
- Some problems have more than one case. One way to solve such problems is to establish cases that together cover all of the possibilities. Calculate the number of arrangements for each case and then add the values for all cases to obtain the total number of arrangements.

Check Your Understanding

Practise

1. Use an organized list or a tree diagram to identify the possible arrangements for

a) the ways that three friends, Jo, Amy, and Mike, can arrange themselves in a row.

b) the ways that you can arrange the digits 2, 5, 8, and 9 to form two-digit numbers.

c) the ways that a customer can choose a starter, a main course, and a dessert from the following menu.

LUNCH SPECIAL MENU
Starter: *soup or salad*
Main: *chili or hamburger or chicken or fish*
Dessert: *ice cream or fruit salad*

2. Evaluate each expression.

a) ${}_8P_2$ **b)** ${}_7P_5$

c) ${}_6P_6$ **d)** ${}_4P_1$

3. Show that $4! + 3! \neq (4 + 3)!$.

4. What is the value of each expression?

a) $9!$ **b)** $\frac{9!}{5!4!}$

c) $(5!)(3!)$ **d)** $6(4!)$

e) $\frac{102!}{100!2!}$ **f)** $7! - 5!$

5. In how many different ways can you arrange all of the letters of each word?

a) hoodie **b)** decided

c) aqilluqqaaq **d)** deeded

e) puppy **f)** baguette

Did You Know?

The Inuit have many words to describe snow. The word *aqilluqqaaq* means fresh and soggy snow in one dialect of Inuktitut.

6. Four students are running in an election for class representative on the student council. In how many different ways can the four names be listed on the ballot?

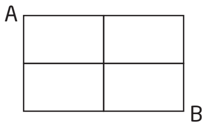
7. Solve for the variable.

a) ${}_nP_2 = 30$ b) ${}_nP_3 = 990$

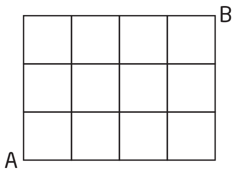
c) ${}_6P_r = 30$ d) $2({}_nP_2) = 60$

8. Determine the number of pathways from A to B.

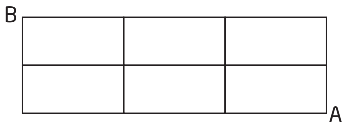
a) Move only down or to the right.



b) Move only up or to the right.



c) Move only up or to the left.



9. Describe the cases you could use to solve each problem. Do not solve.

a) How many 3-digit even numbers greater than 200 can you make using the digits 1, 2, 3, 4, and 5?

b) How many four-letter arrangements beginning with either B or E and ending with a vowel can you make using the letters A, B, C, E, U, and G?

10. In how many ways can four girls and two boys be arranged in a row if

a) the boys are on each end of the row?

b) the boys must be together?

c) the boys must be together in the middle of the row?

11. In how many ways can seven books be arranged on a shelf if

a) the books are all different?

b) two of the books are identical?

c) the books are different and the mathematics book must be on an end?

d) the books are different and four particular books must be together?

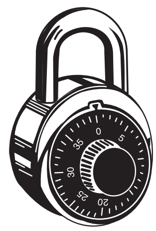
Apply

12. How many six-letter arrangements can you make using all of the letters A, B, C, D, E, and F, without repetition? Of these, how many begin and end with a consonant?

13. A national organization plans to issue its members a 4-character ID code. The first character can be any letter other than O. The last 3 characters are to be 3 different digits. If the organization has 25 300 members, will they be able to assign each member a different ID code? Explain.

14. Iblauk lives in Baker Lake, Nunavut. She makes oven mitts to sell. She has wool duffel in red, dark blue, green, light blue, and yellow for the body of each mitt. She has material for the wrist edge in dark green, pink, royal blue, and red. How many different colour combinations of mitts can Iblauk make?

15. You have forgotten the number sequence to your lock. You know that the correct code is made up of three numbers (right-left-right). The numbers can be from 0 to 39 and repetitions are allowed. If you can test one number sequence every 15 s, how long will it take to test all possible number sequences? Express your answer in hours.



16. Jodi is parking seven different types of vehicles side by side facing the display window at the dealership where she works.

a) In how many ways can she park the vehicles?

b) In how many ways can she park them so that the pickup truck is next to the hybrid car?

c) In how many ways can she park them so that the convertible is not next to the subcompact?

17. a) How many arrangements using all of the letters of the word *parallel* are possible?

b) How many of these arrangements have all of the *l*'s together?

18. The number of different permutations using all of the letters in a particular set is given by $\frac{5!}{2!2!}$.

a) Create a set of letters for which this is true.

b) What English word could have this number of arrangements of its letters?

19. How many integers from 3000 to 8999, inclusive, contain no 7s?

20. Postal codes in Canada consist of three letters and three digits. Letters and digits alternate, as in the code R7B 5K1.

a) How many different postal codes are possible with this format?

b) Do you think Canada will run out of postal codes? Why or why not?

Did You Know?

The Canadian postal code system was established in 1971. The first letters of the codes are assigned to provinces and territories from east to west:

A = Newfoundland and Labrador

...

Y = Yukon Territory

Some provinces have more than one letter, such as H and J for Québec. Some letters, such as I, are not currently used.

21. *Cent mille milliards de poèmes* (*One Hundred Million Million Poems*) was written in 1961 by Raymond Queneau, a French poet, novelist, and publisher. The book is 10 pages long, with 1 sonnet per page. A sonnet is a poem with 14 lines. Each line of every sonnet can be replaced by a line at the same position on a different page. Regardless of which lines are used, the poem makes sense.

a) How many arrangements of the lines are possible for one sonnet?

b) Is the title of the book of poems reasonable? Explain.

22. Use your understanding of factorial notation and the symbol ${}_nP_r$ to solve each equation.

a) ${}_3P_r = 3!$

b) ${}_7P_r = 7!$

c) ${}_nP_3 = 4({}_{n-1}P_2)$

d) $n({}_5P_3) = {}_7P_5$

23. Use ${}_nP_n$ to show that $0! = 1$.

24. Explain why ${}_3P_5$ gives an error message when evaluated on a calculator.

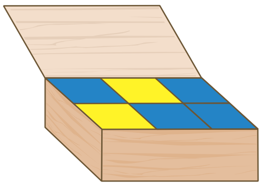
25. How many odd numbers of at most three digits can be formed using the digits 0, 1, 2, 3, 4, and 5 without repetitions?

26. How many even numbers of at least four digits can be formed using the digits 0, 1, 2, 3, and 5 without repetitions?

27. How many integers between 1 and 1000 do not contain repeated digits?

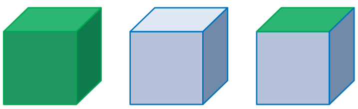
28. A box with a lid has inside dimensions of 3 cm by 2 cm by 1 cm. You have four identical blue cubes and two identical yellow cubes, each 1 cm by 1 cm by 1 cm.

How many different six-cube arrangements of blue and yellow cubes are possible? You must be able to close the lid after any arrangement. The diagram below shows one possible arrangement. Show two different ways to solve the problem.



Extend

29. You have two colours of paint. In how many different ways can you paint the faces of a cube if each face is painted? Painted cubes are considered to be the same if you can rotate one cube so that it matches the other one exactly.



30. Nine students take a walk on four consecutive days. They always walk in rows of three across. Show how to arrange the students so that each student walks only once in a row with any two other students during the four-day time frame. In other words, no three-across triplets are repeated.

> **Did You Know?**
>
> Thomas Kirkman (1806–1895) was born in England and studied mathematics in Dublin. He first presented a version of the problem in #30 in 1847 in the *Cambridge and Dublin Mathematical Journal*. Subsequently, it was published as the "fifteen schoolgirl problem" in the 1850 *Ladies' and Gentlemen's Diary*. There are many solutions and generalizations of the problem.

31. If 100! is evaluated, how many zeros are at the end of the number? Explain how you know.

32. There are five people: A, B, C, D, and E.

The following pairs know each other: A and C, B and C, A and D, D and E, and C and D.

a) Arrange the five people in a row so that nobody is next to a stranger.

b) How many different arrangements are possible such that nobody is next to a stranger?

c) The five people are joined by a sixth person, F, who knows only A. In how many ways can the six people stand in a row if nobody can be next to a stranger? Explain your answer.

Create Connections

C1 **a)** Explain what the notation ${}_aP_b$ represents. Use examples to support your explanation.

b) Which statement best describes the relationship between b and a? Explain.

$b > a \quad b = a \quad b < a \quad b \leq a \quad b \geq a$

C2 Explain why a set of n objects, a of which are of one type and b of which are of a second type, can be arranged in $\dfrac{n!}{a!b!}$ different ways and not in $n!$ ways.

C3 Simplify.

a) $\dfrac{3!(n+2)!}{4!(n-1)!}$

b) $\dfrac{7!(r-1)!}{6!(r+1)!} + \dfrac{5!r!}{3!(r+1)!}$

C4 Create a mathematics career file for this chapter. Identify one occupation or career requiring the use of, or connections to, the mathematics in this section. Write at least two problems that might be used by someone working in the chosen occupation or career. Briefly describe how your problems relate to the occupation or career.

C5 **a)** What is the value of 9!?

b) Determine the value of log (9!).

c) Determine the value of log (10!).

d) How are the answers to parts b) and c) related? Explain why.

11.2

Combinations

Sorting by hand mail that has been rejected by the machine sort due to unrecognizable hand-written or missing postal codes

Focus on...

- explaining the differences between a permutation and a combination
- determining the number of ways to select r elements from n different elements
- solving problems using the number of combinations of n different elements taken r at a time
- solving an equation that involves $_nC_r$ notation

Sometimes you must consider the order in which the elements of a set are arranged. In other situations, the order is not important. For example, when addressing an envelope, it is important to write the six-character postal code in the correct order. In contrast, addressing an envelope, affixing a stamp, and inserting the contents can be completed in any order.

In this section, you will learn about counting outcomes when order does not matter.

Did You Know?

In the six-character postal code used in Canada, the first three characters define a geographical region and the last three characters specify a local delivery unit.

Investigate Making Selections When Order Is Not Important

Problem solving, reasoning, and decision-making are highly prized skills in today's workforce. Here is your opportunity to demonstrate those skills.

1. From a group of four students, three are to be elected to an executive committee with a specific position. The positions are as follows:

1st position	President
2nd position	Vice President
3rd position	Treasurer

a) Does the order in which the students are elected matter? Why?

b) In how many ways can the positions be filled from this group?

2. Now suppose that the three students are to be selected to serve on a committee.

a) Is the order in which the three students are selected still important? Why or why not?

b) How many committees from the group of four students are now possible?

c) How does your answer in part b) relate to the answer in step 1b)?

3. You are part of a group of 6 students.

a) How many handshakes are possible if each student shakes every other student's hand once?

b) What strategies could you use to solve this problem? Discuss with a partner and try to solve the problem in more than one way.

Reflect and Respond

4. What formula could you create to solve a handshake problem involving n students?

5. In step 1, you worked with permutations, but in step 2, you worked with **combinations**. Identify all of the possible three-letter permutations and three-letter combinations of the letters A, B, and C. What are the similarities between permutations and combinations? What are the differences?

combination

- a selection of objects without regard to order
- all of the three-letter combinations of P, Q, R, and S are PQR, PQS, PRS, and QRS (arrangements such as PQR and RPQ are the same combination)

Link the Ideas

A combination is a selection of a group of objects, taken from a larger group, for which the kinds of objects selected is important, but not the order in which they are selected.

There are several ways to find the number of possible combinations. One is to use reasoning. Use the fundamental counting principle and divide by the number of ways that the objects can be arranged among themselves. For example, calculate the number of combinations of three digits made from the digits 1, 2, 3, 4, and 5 without repetitions:

Number of choices for the first digit	Number of choices for the second digit	Number of choices for the third digit
5	4	3

There are 5 × 4 × 3 or 60 ways to arrange 3 items from 5. However, 3 digits can be arranged in 3! ways among themselves, and in a combination these are considered to be the same selection.

So,

$$\text{number of combinations} = \frac{\text{number of permutations}}{3!} = \frac{60}{3!} = \frac{60}{6} = 10$$

What does 3! represent?

The notation ${}_nC_r$, or $\binom{n}{r}$, represents the number of combinations of n items taken r at a time, where $n \geq r$ and $r \geq 0$.

$$\begin{aligned} {}_nC_r &= \frac{{}_nP_r}{r!} \\ &= \frac{\frac{n!}{(n-r)!}}{r!} \\ &= \frac{n!}{(n-r)!r!} \end{aligned}$$

Why must $n \geq r \geq 0$?

Did You Know?

The number of combinations of n items taken r at a time is equivalent to the number of combinations of n items taken $n - r$ at a time.

${}_nC_r = {}_nC_{n-r}$

The number of ways of choosing three digits from five digits is

$$\begin{aligned} {}_5C_3 &= \frac{5!}{(5-3)!3!} \\ &= \frac{5!}{2!3!} \\ &= \frac{(5)(4)}{(2)(1)} \\ &= 10 \end{aligned}$$

Explain how to simplify the expression in step 2 to get the expression shown in step 3.

How many ways are there to choose two digits from five digits? What do you notice?

There are ten ways to select three items from a set of five.

Example 1

Combinations and the Fundamental Counting Principle

There are 12 females and 18 males in a grade 12 class. The principal wishes to meet with a group of 5 students to discuss graduation.

a) How many selections are possible?

b) How many selections are possible if the group consists of two females and three males?

c) One of the female students is named Brooklyn. How many five-member selections consisting of Brooklyn, one other female, and three males are possible?

Solution

Ask yourself if the order of selection is important in these questions.

a) The question involves choosing 5 students out of 30. In this group, the order of selection is unimportant. So, this is a combinations problem. Use the combinations formula.

Substitute $n = 30$ and $r = 5$ into ${}_nC_r = \dfrac{n!}{(n - r)!r!}$:

$$\begin{aligned}
{}_{30}C_5 &= \frac{30!}{(30 - 5)!5!} \\
&= \frac{30!}{25!5!} \\
&= \frac{(\cancel{30}^{1})(29)(\cancel{28}^{7})(27)(26)(\cancel{25!}^{1})}{\cancel{25!}_{1}(\cancel{5}_{1})(\cancel{4}_{1})(\cancel{3}_{1})(\cancel{2}_{1})(1)} \\
&= 142\ 506
\end{aligned}$$

There are 142 506 possible ways of selecting the group of 5 students.

b) There are ${}_{12}C_2$ ways of selecting two female students.

There are ${}_{18}C_3$ ways of selecting three male students.

Using the fundamental counting principle, the number of ways of selecting two females and three males is

$$\begin{aligned}
{}_{12}C_2 \times {}_{18}C_3 &= \frac{12!}{(12 - 2)!2!} \times \frac{18!}{(18 - 3)!3!} \\
&= \frac{(\cancel{12}^{6})(11)(\cancel{10!}^{1})}{(\cancel{10!}_{1})(\cancel{2}_{1})(1)} \times \frac{(\cancel{18}^{3})(17)(16)(\cancel{15!}^{1})}{(\cancel{15!}_{1})(\cancel{3}_{1})(\cancel{2}_{1})(1)} \\
&= 66 \times 816 \\
&= 53\ 856
\end{aligned}$$

Why are the elements ${}_{12}C_2$ and ${}_{18}C_3$ multiplied together?

There are 53 856 ways to select a group consisting of 2 females and 3 males.

c) There is one way to select Brooklyn.

There are 11 females remaining, so there are ${}_{11}C_1$ or 11 choices for the second female.

Why is ${}_{11}C_1 = 11$?

There are ${}_{18}C_3$ ways to select the three males.

There are $1 \times 11 \times {}_{18}C_3$ or 8976 ways to select this five-member group.

Your Turn

In how many ways can the debating club coach select a team from six grade 11 students and seven grade 12 students if the team has

a) four members?

b) four members, only one of whom is in grade 11?

Example 2

Combinations With Cases

Rianna is writing a geography exam. The instructions say that she must answer a specified number of questions from each section. How many different selections of questions are possible if

a) she must answer two of the four questions in part A and three of the five questions in part B?

b) she must answer two of the four questions in part A and at least four of the five questions in part B?

Solution

Why should you use combinations rather than permutations to solve this problem?

a) The number of ways of selecting two questions in part A is ${}_4C_2$.

The number of ways of selecting three questions in part B is ${}_5C_3$.

According to the fundamental counting principle, the number of possible question selections is ${}_4C_2 \times {}_5C_3 = 6 \times 10$ or 60.

There are 60 different ways in which Rianna can choose 2 of the 4 questions in part A and 3 of the 5 questions in part B.

b) "At least four" means that Rianna can answer either four questions or five questions in part B. Solve the problem using two cases.

Case 1: Answering Four Questions in Part B

Part A Choices	Part B Choices
${}_4C_2$	${}_5C_4$

The number of ways of choosing these questions is ${}_4C_2 \times {}_5C_4 = 6 \times 5$ or 30.

Why do you multiply the possibilities for parts A and B?

Case 2: Answering Five Questions in Part B

Part A Choices	Part B Choices
${}_4C_2$	${}_5C_5$

The number of ways of choosing these questions is ${}_4C_2 \times {}_5C_5 = 6 \times 1$ or 6.

Each case represents an exclusive or separate event. The final answer is the sum of both cases.

The number of possible ways of choosing either 4 questions or 5 questions in part B is $30 + 6$ or 36.

Why do you add the two cases?

Your Turn

A bag contains seven black balls and six red balls. In how many ways can you draw groups of five balls if at least three must be red?

Example 3

Simplifying Expressions and Solving Equations With Combinations

a) Express as factorials and simplify $\dfrac{{}_nC_5}{{}_{n-1}C_3}$.

b) Solve for n if $2({}_nC_2) = {}_{n+1}C_3$.

Solution

a)
$$\frac{{}_nC_5}{{}_{n-1}C_3} = \frac{\dfrac{n!}{(n-5)!5!}}{\dfrac{(n-1)!}{(n-4)!3!}}$$

What is the formula for ${}_nC_r$?

Why is $(n-4)!$ in the lower denominator?

$$= \left(\frac{n!}{(n-5)!5!}\right)\left(\frac{(n-4)!3!}{(n-1)!}\right)$$

$$= \frac{n\cancel{(n-1)!}}{\cancel{(n-5)!}(5)(4)\cancel{(3!)}} \times \frac{(n-4)\cancel{(n-5)!}\cancel{3!}}{\cancel{(n-1)!}}$$

Explain why $n!$ can be written as $n(n-1)!$.

$$= \frac{n(n-4)}{20}$$

b)
$$2({}_nC_2) = {}_{n+1}C_3$$
$$\cancel{2}\left(\frac{n!}{\cancel{(n-2)!}\cancel{2!}}\right) = \frac{(n+1)!}{\cancel{(n-2)!}3!}$$
$$n! = \frac{(n+1)!}{3!}$$
$$3! = \frac{(n+1)!}{n!}$$
$$6 = \frac{(n+1)\cancel{(n!)}}{\cancel{n!}}$$
$$6 = n + 1$$
$$5 = n$$

Your Turn

a) Express in factorial notation and simplify $({}_{n-1}C_3)\left(\dfrac{1}{{}_{n-2}C_3}\right)$.

b) Solve for n if $720({}_nC_5) = {}_{n+1}P_5$.

Key Ideas

- A selection of objects in which order is not important is a combination.
- When determining the number of possibilities in a situation, if order matters, it is a permutation. If order does not matter, it is a combination.
- The number of combinations of n objects taken r at a time can be represented by ${}_nC_r$, where $n \geq r$ and $r \geq 0$. A formula for ${}_nC_r$ is ${}_nC_r = \dfrac{{}_nP_r}{r!}$ or ${}_nC_r = \dfrac{n!}{(n-r)!r!}$.

Check Your Understanding

The questions in this section involve permutations or combinations. Always determine whether order is important.

Practise

1. Decide whether each of the following is a combination or a permutation problem. Briefly describe why. You do not need to solve the problem.

a) In a traditional Aboriginal welcome circle, each member shakes hands with each other member twice. If there are eight people in a welcome circle, how many handshakes occur?

b) How many numbers less than 300 can you make using the digits 1, 2, 3, 4, and 5?

c) A car dealer has 15 mid-sized cars. In how many ways can a rental agency purchase 10 of the cars?

d) A hockey team has 18 players. In how many ways can the driver select six of the players to ride in the team van?

2. Describe the differences between ${}_5P_3$ and ${}_5C_3$, and then evaluate each one.

3. Evaluate.

a) ${}_6P_4$ **b)** ${}_7C_3$

c) ${}_5C_2$ **d)** ${}_{10}C_7$

4. From ten employees, in how many ways can you

a) select a group of four?

b) assign four different jobs?

5. a) List all of the combinations of A, B, C, and D taken two at a time.

b) List all of the permutations of A, B, C, and D taken two at a time.

c) How is the number of combinations related to the number of permutations?

6. Solve for n.

a) ${}_nC_1 = 10$ **b)** ${}_nC_2 = 21$

c) ${}_nC_{n-2} = 6$ **d)** ${}_{n+1}C_{n-1} = 15$

7. Identify the cases you could use to solve each problem. Do not solve.

a) How many numbers less than 1000 can you make using any number of the digits 1, 2, 3, 4, and 5?

b) In how many ways can a team be selected from six grade 11 students and five grade 12 students if the five-person team has four members from either grade and a spare from grade 11?

8. Show that ${}_{11}C_3 = {}_{11}C_8$.

9. a) Evaluate ${}_5C_5$ to determine the number of ways you can select five objects from a group of five.

b) Evaluate ${}_5C_0$ to determine the number of ways you can select no objects from a group of five. Explain why the answer makes sense.

Apply

10. From a penny, a nickel, a dime, and a quarter, how many different sums of money can be formed consisting of

a) three coins?

b) at most two coins?

11. From six females, in how many ways can you select

a) a group of four females?

b) a group of at least four females?

12. Verify the identity ${}_nC_{r-1} + {}_nC_r = {}_{n+1}C_r$.

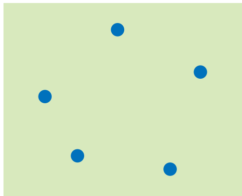
13. At the local drive-in, you can order a burger with tomato, lettuce, pickle, hot peppers, onion, or cheese. How many different burgers with any three different choices for the extras can you order? Does this question involve permutations or combinations? Explain.

14. A pizzeria offers ten different toppings.

a) How many different four-topping pizzas are possible?

b) Is this a permutation or a combination question? Explain.

15. Consider five points, no three of which are collinear.



a) How many line segments can you draw connecting any two of the points? Complete this question using two different methods.

b) How many triangles, with vertices selected from the given points, can you draw?

c) Write an expression using factorials for the number of triangles if there are ten non-collinear points. How does this answer compare to the number of line segments for the same ten points?

16. Verify that ${}_nC_r = {}_nC_{n-r}$.

17. A jury pool consists of 12 women and 8 men.

a) How many 12-person juries can be selected?

b) How many juries containing seven women and five men can be selected?

c) How many juries containing at least ten women can be selected?

18. Consider a standard deck of 52 well-shuffled cards.

a) In how many ways can you select five cards?

b) In how many ways can you select five cards if three of them are hearts?

c) In how many ways can you select five cards if only one of them is black?

Did You Know?

A standard deck of playing cards contains 52 cards in four suits: clubs, diamonds, hearts, and spades. Each suit contains 13 cards labelled 2 to 10, jack, queen, king, and ace. Playing cards are thought to have originated in India. They were introduced into Europe around 1275.

19. a) In how many ways can you select a set of four science books and three geography books from six different science books and seven different geography books?

b) In how many ways can you place the four science books and the three geography books in a row on a shelf if the science books must remain together?

20. A Manitoba gallery wishes to display 20 paintings to showcase the work of artist George Fagnan.

a) How many selections are possible if the artist allows the gallery to choose from 40 of his works? Leave your answer in factorial form.

b) The gallery curator wants to set 4 of the paintings from the 20 selected in a row near the entrance. In how many ways can this be accomplished?

Did You Know?

George Fagnan grew up in Swan River, Manitoba. He is a proud member of the Sapotaweyak Cree Nation, and he currently lives in Brandon, Manitoba. He began his art career around the age of 5. He enjoys traditional native art and other creative activities.

Blue Garden, 2009

21. The cards from a standard deck of playing cards are dealt to 4 people, 13 cards at a time. This means that the first person receives the first 13 cards, the second person gets the next 13 cards, and so on.

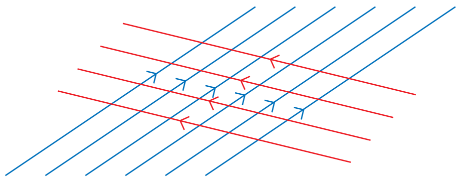
a) How many such sets of four 13-card hands can be dealt? Leave your answer as a product of factorials.

b) Without using a calculator, show that the answer in part a) simplifies to $\frac{52!}{(13!)^4}$.

c) Evaluate the answer to part a).

Extend

22. How many parallelograms are formed if four parallel lines intersect another set of six parallel lines? The lines in the first set are not parallel to the lines in the second set.



23. In a bowl of ice cream, the order of the scoops does not matter.

a) Suppose you can make 630 two-scoop bowls of ice cream, each containing two different flavours, at the shop where you work. How many flavours of ice cream are available in this shop?

b) How many two-scoop bowls could you make if you can duplicate flavours?

24. Consider the following conjecture. If p is a prime number, ${}_aC_b$ and ${}_{pa}C_{pb}$ have the same remainder when you divide by p.

a) Show that the statement is true for ${}_5C_2$ when $p = 3$.

b) Is this statement true for ${}_5C_2$ when $p = 7$? What is the remainder?

c) How many remainders are possible when dividing by 7? What are they?

d) Describe what you could do to prove the initial conjecture.

Create Connections

C1 Does a combination lock involve combinations in a mathematical sense? Explain.

C2 **a)** Explain what the notation ${}_aC_b$ represents. Use examples to support your explanation.

b) Write an inequality that describes the relationship between all possible values for a and b.

c) What can you say for sure about the value of b?

C3 A teacher asks her students to calculate the number of ways in which a hospital administrator could assign four patients to six private rooms. Beth says that the answer is ${}_6C_4$. Bryan disagrees. He claims the answer is ${}_6P_4$. Who is correct? Why?

C4 MINI LAB Eight points lie on the circumference of a circle. Explore how many different inscribed quadrilaterals can be drawn using the points as vertices.

Step 1 Suppose the eight points are on the unit circle at P(0°), P(45°), P(90°), P(135°), P(180°), P(225°), P(270°), and P(315°). Draw a diagram. Show a quadrilateral that is an isosceles trapezoid with four of the given points as vertices.

Step 2 Create a table in which you identify the number of possible quadrilaterals that are squares, rectangles, parallelograms, and isosceles trapezoids that can be created using four of the eight points from Step 1.

Step 3 Make a conclusion. How many different inscribed quadrilaterals can be drawn using four of the eight points that lie on the circumference of a circle as vertices?

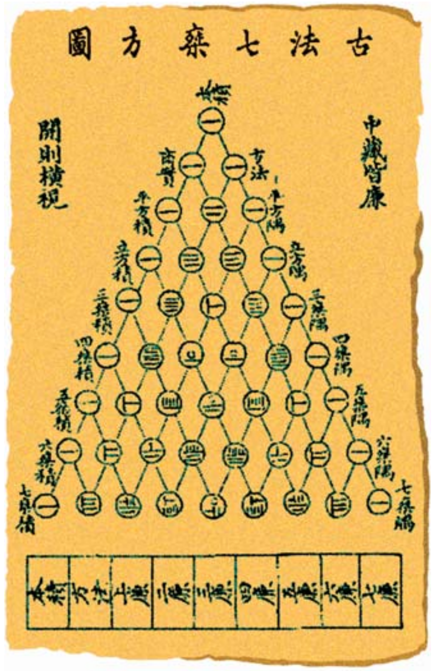
11.3

The Binomial Theorem

Focus on...

- relating the coefficients in the expansion of $(x + y)^n$, $n \in N$, to Pascal's triangle and to combinations
- expanding $(x + y)^n$, $n \in N$, in a variety of ways, including the binomial theorem
- determining a specific term in the expansion of $(x + y)^n$

In 1653, Blaise Pascal, a French mathematician, described a triangular array of numbers corresponding to the number of ways to choose r elements from a set of n objects. Some interesting number patterns occur in Pascal's triangle. Have you encountered Pascal's triangle before? Have you explored its many patterns? Did you realize it can give you the number of combinations in certain situations?



Yang Hui's triangle, 13th century China

Investigate Patterns in Pascal's Triangle

Materials

- counters
- copy of Pascal's triangle

1. Examine Pascal's triangle and identify at least three patterns. Compare and discuss your patterns with a partner.

```
            1
         1     1
      1     2     1
   1     3     3     1
1     4     6     4     1
```

2. Write the next row for the Pascal's triangle shown.

3. Some of the patterns in Pascal's triangle are spatial and relate to whole sections in the chart. Create a large Pascal's triangle with at least 20 rows. Mark or use counters to cover all of the multiples of 7 in your 20-row triangle. Then, cover all of the multiples of 5 and multiples of 3. What do you conclude? What happens for multiples of even numbers?

4. Other patterns may appear unexpectedly. Determine the sum of the numbers in each horizontal row. What pattern did you find?

5. Each number in Pascal's triangle can be written as a combination using the notation ${}_nC_r$, where n is the number of objects in the set and r is the number selected. For example, you can express the third row as

${}_2C_0 \quad {}_2C_1 \quad {}_2C_2$

Express the fifth row using combination notation. Check whether your combinations have the same values as the numbers in the fifth row of Pascal's triangle.

Did You Know?

Pascal was not the first person to discover the triangle of numbers that bears his name. It was known in India, Persia, and China centuries before. The Chinese called it "Yang Hui's triangle" in honour of Yang Hui, who lived from 1238 to 1298.

6. Expand the following binomials by multiplying.
   $(x + y)^2$
   $(x + y)^3$
   $(x + y)^4$

Reflect and Respond

7. Explain how to get the numbers in the next row from the numbers in the previous row of Pascal's triangle. Use examples.
8. How are the values you obtained in steps 4 and 5 related? Explain using values from specific rows.
9. How do the coefficients of the simplified terms in your binomial expansions in step 6 relate to Pascal's triangle?

Link the Ideas

If you expand a power of a binomial expression, you get a series of terms.

$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$

How could you get this expansion by multiplying?

There are many patterns in the binomial expansion of $(x + y)^4$.

What patterns do you observe?

The coefficients in a binomial expansion can be determined from Pascal's triangle. In the expansion of $(x + y)^n$, where $n \in N$, the coefficients of the terms are identical to the numbers in the $(n + 1)$th row of Pascal's triangle.

Binomial	Pascal's Triangle in Binomial Expansion	Row
$(x + y)^0$	1	1
$(x + y)^1$	$1x + 1y$	2
$(x + y)^2$	$1x^2 + 2xy + 1y^2$	3
$(x + y)^3$	$1x^3 + 3x^2y + 3xy^2 + 1y^3$	4
$(x + y)^4$	$1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$	5

The coefficients in a binomial expansion can also be determined using combinations.

Pascal's Triangle	Combinations
1	$_0C_0$
1 1	$_1C_0$ $_1C_1$
1 2 1	$_2C_0$ $_2C_1$ $_2C_2$
1 3 3 1	$_3C_0$ $_3C_1$ $_3C_2$ $_3C_3$
1 4 6 4 1	$_4C_0$ $_4C_1$ $_4C_2$ $_4C_3$ $_4C_4$
1 5 10 10 5 1	$_5C_0$ $_5C_1$ $_5C_2$ $_5C_3$ $_5C_4$ $_5C_5$

$$\begin{aligned} {}_5C_2 &= \frac{5!}{3!2!} \\ &= \frac{(5)(4)}{2} \\ &= 10 \end{aligned}$$

Note that $_5C_2$ represents the number of combinations of five items taken two at a time. In the expansion of $(x + y)^5$, it represents the coefficient of the term containing x^3y^2 and shows the number of selections possible for three x's and two y's.

Example 1

Expand Binomials

a) Expand $(p + q)^6$.
b) Identify patterns in the expansion of $(p + q)^6$.

Solution

a) Method 1: Use Patterns and Pascal's Triangle

The coefficients for the terms of the expansion of $(p + q)^6$ occur in the $(6 + 1)$th or seventh row of Pascal's triangle.

Why is the row number different by one from the exponent on the binomial?

The seventh row of Pascal's triangle is
1 6 15 20 15 6 1

How are the numbers obtained?

$$(p + q)^6 = 1(p)^6(q)^0 + 6(p)^5(q)^1 + 15(p)^4(q)^2 + 20(p)^3(q)^3 + 15(p)^2(q)^4 + 6(p)^1(q)^5 + 1(p)^0(q)^6$$
$$= p^6 + 6p^5q + 15p^4q^2 + 20p^3q^3 + 15p^2q^4 + 6pq^5 + q^6$$

Method 2: Use Combinations to Determine Coefficients in the Expansion

$$(p + q)^6 = {}_6C_0(p)^6(q)^0 + {}_6C_1(p)^5(q)^1 + {}_6C_2(p)^4(q)^2 + {}_6C_3(p)^3(q)^3 + {}_6C_4(p)^2(q)^4 + {}_6C_5(p)^1(q)^5 + {}_6C_6(p)^0(q)^6$$
$$= p^6 + 6p^5q + 15p^4q^2 + 20p^3q^3 + 15p^2q^4 + 6pq^5 + q^6$$

Show that ${}_6C_4 = 15$. How does symmetry help you find the terms?

b) Some patterns are as follows:
- There are 6 + 1, or 7, terms in the expansion of $(p + q)^6$.
- The powers of p decrease from 6 to 0 in successive terms of the expansion.
- The powers of q increase from 0 to 6.
- Each term is of degree 6 (the sum of the exponents for p and q is 6 for each term)
- The coefficients are symmetrical, 1 6 15 20 15 6 1, and begin and end with 1.

Your Turn

a) What are the coefficients in the expansion of $(c + d)^5$?
b) Do you prefer to use Pascal's triangle or combinations to determine the coefficients in a binomial expansion? Why?
c) How many terms are in the expansion of $(c + d)^5$?
d) What is the simplified expression for the second term in the expansion of $(c + d)^5$ if the terms are written with descending powers of c?

binomial theorem

- used to expand $(x + y)^n$, $n \in \mathrm{N}$
- each term has the form $_nC_k(x)^{n-k}(y)^k$, where $k + 1$ is the term number

You can use the **binomial theorem** to expand any power of a binomial expression.

$$(x + y)^n = {}_nC_0(x)^n(y)^0 + {}_nC_1(x)^{n-1}(y)^1 + {}_nC_2(x)^{n-2}(y)^2 + \cdots + {}_nC_{n-1}(x)^1(y)^{n-1} + {}_nC_n(x)^0(y)^n$$

In this chapter, all binomial expansions will be written in descending order of the exponent of the first term in the binomial.

The following are some important observations about the expansion of $(x + y)^n$, where x and y represent the terms of the binomial and $n \in \mathrm{N}$:

- the expansion contains $n + 1$ terms
- the number of objects, k, selected in the combination $_nC_k$ can be taken to match the number of factors of the second variable selected; that is, it is the same as the exponent on the second variable
- the general term, t_{k+1}, has the form

$$_nC_k(x)^{n-k}(y)^k$$

the same

- the sum of the exponents in any term of the expansion is n

Did You Know?

In French, the binomial theorem is referred to as Newton's binomial formula (binôme de Newton). While Newton was not the first to describe binomial expansion, he did develop a formula that can be used to expand the general case $(x + y)^n$, $n \in \mathrm{R}$.

Example 2

Use the Binomial Theorem

a) Use the binomial theorem to expand $(2a - 3b)^4$.

b) What is the third term in the expansion of $(4b - 5)^6$?

c) In the expansion of $\left(a^2 - \dfrac{1}{a}\right)^5$, which term, in simplified form, contains a? Determine the value of the term.

Solution

a) Use the binomial theorem to expand $(x + y)^n$, $n \in \mathrm{N}$.

$$(x + y)^n = {}_nC_0(x)^n(y)^0 + {}_nC_1(x)^{n-1}(y)^1 + {}_nC_2(x)^{n-2}(y)^2 + \cdots + {}_nC_{n-1}(x)^1(y)^{n-1} + {}_nC_n(x)^0(y)^n$$

In this case, $(2a - 3b)^4 = [2a + (-3b)]^4$, so, in the binomial expansion, substitute $x = 2a$, $y = -3b$, and $n = 4$.

$$\begin{aligned}
&(2a - 3b)^4 \\
&= {}_4C_0(2a)^4(-3b)^0 + {}_4C_1(2a)^3(-3b)^1 + {}_4C_2(2a)^2(-3b)^2 + {}_4C_3(2a)^1(-3b)^3 \\
&\quad + {}_4C_4(2a)^0(-3b)^4 \\
&= 1(16a^4)(1) + 4(8a^3)(-3b) + 6(4a^2)(9b^2) + 4(2a)(-27b^3) + 1(1)(81b^4) \\
&= 16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4
\end{aligned}$$

What pattern occurs in the signs of the terms?

b) The coefficients in the expansion of $(4b - 5)^6$ involve the pattern ${}_6C_0$, ${}_6C_1$, ${}_6C_2$, ${}_6C_3$, ….

The coefficient of the third term involves ${}_6C_2$.

Why does the coefficient of the third term not involve ${}_6C_3$?

In the general term $t_{k+1} = {}_nC_k(x)^{n-k}(y)^k$, substitute $x = 4b$, $y = -5$, $n = 6$, and $k = 2$.

$$\begin{aligned} t_3 &= {}_6C_2(4b)^{6-2}(-5)^2 \\ &= \frac{6!}{4!2!}(4b)^4(-5)^2 \\ &= (15)(256b^4)(25) \\ &= 96\ 000b^4 \end{aligned}$$

The third term in the expansion of $(4b - 5)^6$ is $96\ 000b^4$.

c) Determine the first few terms of the expanded binomial. Simplify the variable part of each term to find the pattern.

In the binomial expansion, substitute $x = a^2$, $y = -\frac{1}{a}$, and $n = 5$.

$$\begin{aligned} \left(a^2 - \frac{1}{a}\right)^5 &= {}_5C_0(a^2)^5\left(-\frac{1}{a}\right)^0 + {}_5C_1(a^2)^4\left(-\frac{1}{a}\right)^1 + {}_5C_2(a^2)^3\left(-\frac{1}{a}\right)^2 + \cdots \\ &= {}_5C_0a^{10} + {}_5C_1(a^8)\left(-\frac{1}{a}\right) + {}_5C_2a^6\left(\frac{1}{a^2}\right) + \cdots \\ &= {}_5C_0a^{10} - {}_5C_1a^7 + {}_5C_2a^4 + \cdots \end{aligned}$$

The pattern shows that the exponents for a are decreasing by 3 in each successive term. The next term will contain a^{4-3} or a^1, the term after that will contain a^{1-3} or a^{-2}, and the last term will contain a^{-5}. The fourth term contains a^1, or a, in its simplest form.

$$\begin{aligned} \text{Its value is } {}_5C_3(a^2)^2\left(-\frac{1}{a}\right)^3 &= 10(a^4)\left(-\frac{1}{a^3}\right) \\ &= -10a \end{aligned}$$

Your Turn

a) How many terms are in the expansion of $(2a - 7)^8$?

b) What is the value of the fourth term in the expansion of $(2a - 7)^8$?

c) Use the binomial theorem to find the first four terms of the expansion of $(3a + 2b)^7$.

Key Ideas

- Pascal's triangle has many patterns. For example, each row begins and ends with 1. Each number in the interior of any row is the sum of the two numbers to its left and right in the row above.
- You can use Pascal's triangle or combinations to determine the coefficients in the expansion of $(x + y)^n$, where n is a natural number.
- You can use the binomial theorem to expand any binomial of the form $(x + y)^n$, $n \in N$.
- You can determine any term in the expansion of $(x + y)^n$ using patterns without having to perform the entire expansion. The general term, t_{k+1}, has the form ${}_nC_k(x)^{n-k}(y)^k$.

Check Your Understanding

Practise

1. Some rows from Pascal's triangle are shown. What is the next row in each case?

a) 1 3 3 1

b) 1 7 21 35 35 21 7 1

c) 1 10 45 120 210 252 210 120 45 10 1

2. Express each row of Pascal's triangle using combinations. Leave each term in the form ${}_nC_r$.

a) 1 2 1

b) 1 4 6 4 1

c) 1 7 21 35 35 21 7 1

3. Express each circled term in the given row of Pascal's triangle as a combination.

a) 1 3 3 1

b) 1 6 15 20 15 6 1

c) 1 1

4. How many terms are in the expansion of each expression?

a) $(x - 3y)^4$

b) $(1 + 3t^2)^7$

c) $(a + 6)^q$

5. Use the binomial theorem to expand.

a) $(x + y)^2$

b) $(a + 1)^3$

c) $(1 - p)^4$

6. Expand and simplify using the binomial theorem.

a) $(a + 3b)^3$

b) $(3a - 2b)^5$

c) $(2x - 5)^4$

7. Determine the simplified value of the specified term.

a) the sixth term of $(a + b)^9$

b) the fourth term of $(x - 3y)^6$

c) the seventh term of $(1 - 2t)^{14}$

d) the middle term of $(4x + y)^4$

e) the second-last term of $(3w^2 + 2)^8$

Apply

8. Explain how Pascal's triangle is constructed.

1 — Row 1
1 1 — Row 2
1 2 1 — Row 3
1 3 3 1 — Row 4
1 4 6 4 1
1 5 10 10 5 1

9. a) Determine the sum of the numbers in each of the first five rows in Pascal's triangle.

b) What is an expression for the sum of the numbers in the ninth row of Pascal's triangle?

c) What is a formula for the sum of the numbers in the nth row?

10. Examine the numbers in each "hockey stick" pattern within Pascal's triangle.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1

a) Describe one pattern for the numbers within each hockey stick.

b) Does your pattern work for all possible hockey sticks? Explain.

11. Answer the following questions for $(x + y)^{12}$ without expanding or computing all of its coefficients.

a) How many terms are in the expansion?

b) What is the simplified fourth term in the expansion?

c) For what value of r does ${}_{12}C_r$ give the maximum coefficient? What is that coefficient?

12. Express each expansion in the form $(a + b)^n$, $n \in \mathrm{N}$.

a) ${}_4C_0x^4 + {}_4C_1x^3y + {}_4C_2x^2y^2 + {}_4C_3xy^3 + {}_4C_4y^4$

b) ${}_5C_0 - {}_5C_1y + {}_5C_2y^2 - {}_5C_3y^3 + {}_5C_4y^4 - {}_5C_5y^5$

13. a) Penelope claims that if you read any row in Pascal's triangle as a single number, it can be expressed in the form 11^m, where m is a whole number. Do you agree? Explain.

b) What could m represent?

14. a) Expand $(x + y)^3$ and $(x - y)^3$. How are the expansions different?

b) Show that $(x + y)^3 + (x - y)^3 = 2x(x^2 + 3y^2)$.

c) What is the result for $(x + y)^3 - (x - y)^3$? How do the answers in parts b) and c) compare?

15. You invite five friends for dinner but forget to ask for a reply.

a) What are the possible cases for the number of dinner guests?

b) How many combinations of your friends could come for dinner?

c) How does your answer in part b) relate to Pascal's triangle?

16. a) Draw a tree diagram that depicts tossing a coin three times. Use H to represent a head and T to represent a tail landing face up. List the arrangements of heads (H) and tails (T) by the branches of your tree diagram.

b) Expand $(\mathrm{H} + \mathrm{T})^3$ by multiplying the factors. In the first step write the factors in full. For example, the first term will be HHH. You should have eight different terms. Simplify this arrangement of terms by writing HHH as H^3, and so on. Combine like terms.

c) What does HHH or H^3 represent in both part a) and part b)? Explain what 3HHT or $3\mathrm{H}^2\mathrm{T}$ represents in parts a) and b).

17. Expand and simplify. Use the binomial theorem.

a) $\left(\dfrac{a}{b} + 2\right)^3$

b) $\left(\dfrac{a}{b} - a\right)^4$

c) $\left(1 - \dfrac{x}{2}\right)^6$

d) $\left(2x^2 - \dfrac{1}{x}\right)^4$

18. a) Determine the middle term in the expansion of $(a - 3b^3)^8$.

b) Determine the term containing x^{11} in the expansion of $\left(x^2 - \dfrac{1}{x}\right)^{10}$.

19. a) Determine the constant term in the expansion $\left(x^2 - \dfrac{2}{x}\right)^{12}$.

b) What is the constant term in the expansion of $\left(y - \dfrac{1}{y^2}\right)^{12}$?

20. One term in the expansion of $(2x - m)^7$ is $-15\,120x^4y^3$. Determine m.

21. MINI LAB Some students argue that using Pascal's triangle to find the coefficients in a binomial expansion is only helpful for small powers. What if you could find a pattern that allowed you to write any row in Pascal's triangle?

Work with a partner. Consider the fifth row in Pascal's triangle. Each number is related to the previous number as shown.

$$1 \xrightarrow{\times\frac{4}{1}} 4 \xrightarrow{\times\frac{3}{2}} 6 \xrightarrow{\times\frac{2}{3}} 4 \xrightarrow{\times\frac{1}{4}} 1$$

Step 1 What pattern do you see in the multipliers? Check whether your pattern works for the sixth row:
1 5 10 10 5 1

Step 2 What pattern exists between the row number and the second element in the row?

Step 3 What are the first 2 terms in the 21st row of Pascal's triangle? What are the multipliers for successive terms in row 21?

Extend

22. Five rows of the Leibniz triangle are shown.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & \frac{1}{2} & & \frac{1}{2} & & & \\
 & & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{3} & & \\
 & \frac{1}{4} & & \frac{1}{12} & & \frac{1}{12} & & \frac{1}{4} & \\
\frac{1}{5} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{20} & & \frac{1}{5}
\end{array}
$$

a) In the Leibniz triangle, each entry is the sum of two numbers. However, it is not the same pattern of sums as in Pascal's triangle. Which two numbers are added to get each entry?

b) Write the next two rows in the Leibniz triangle.

c) Describe at least two patterns in the Leibniz triangle.

Did You Know?

Gottfried Wilhelm Leibniz lived in Germany from 1646 to 1716. He was a great mathematician and philosopher. He has been described as the last universal genius. He developed calculus independently of Sir Isaac Newton and was very involved in the invention of mechanical calculators.

23. Show how to expand a trinomial using the binomial theorem. Expand and simplify $(a + b + c)^3$.

24. a) Complete a table in your notebook similar to the one shown, for one to six points.

The table relates the number of points on the circumference of a circle, the number of possible line segments you can make by joining any two of the points, and the number of triangles, quadrilaterals, pentagons, or hexagons formed. Make your own diagrams 3, 4, 5, and 6. Do not include values of zero in your table.

Diagram	Points	Line Segments	Triangles	Quadrilaterals	Pentagons	Hexagons
	1					
	2	1				
	3					
	4					
	5					
	6					

b) Show how the numbers in any row of the table relate to Pascal's triangle.

c) What values would you expect for eight points on a circle?

25. The real number e is the base of natural logarithms. It appears in certain mathematics problems involving growth or decay and is part of Stirling's formula for approximating factorials. One way to calculate e is shown below.

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

a) Determine the approximate value of e using the first five terms of the series shown.

b) How does the approximate value of e change if you use seven terms? eight terms? What do you conclude?

c) What is the value of e on your calculator?

d) Stirling's approximation can be expressed as

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Use Stirling's approximation to estimate 15!, and compare this result with the true value.

e) A more accurate approximation uses the following variation of Stirling's formula:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left(1 + \frac{1}{12n}\right)$$

Use the formula from part d) and the variation to compare estimates for 50!.

Create Connections

C1 Relate the coefficients of the terms of the expansion of $(x + y)^n$, $n \in N$, to Pascal's triangle. Use at least two examples.

C2 **a)** Create three problems for which $\frac{4!}{2!2!}$ either is an expression for the answer or is part of the answer. One of your problems must be a permutation, one must be a combination, and one must involve the expansion of a power of the binomial $a + b$.

b) Show how your three problems are similar and how they are different.

C3 **a)** Which method, Pascal's triangle or combinations, do you prefer to use to express the coefficients in the expansion of $(a + b)^n$, $n \in N$?

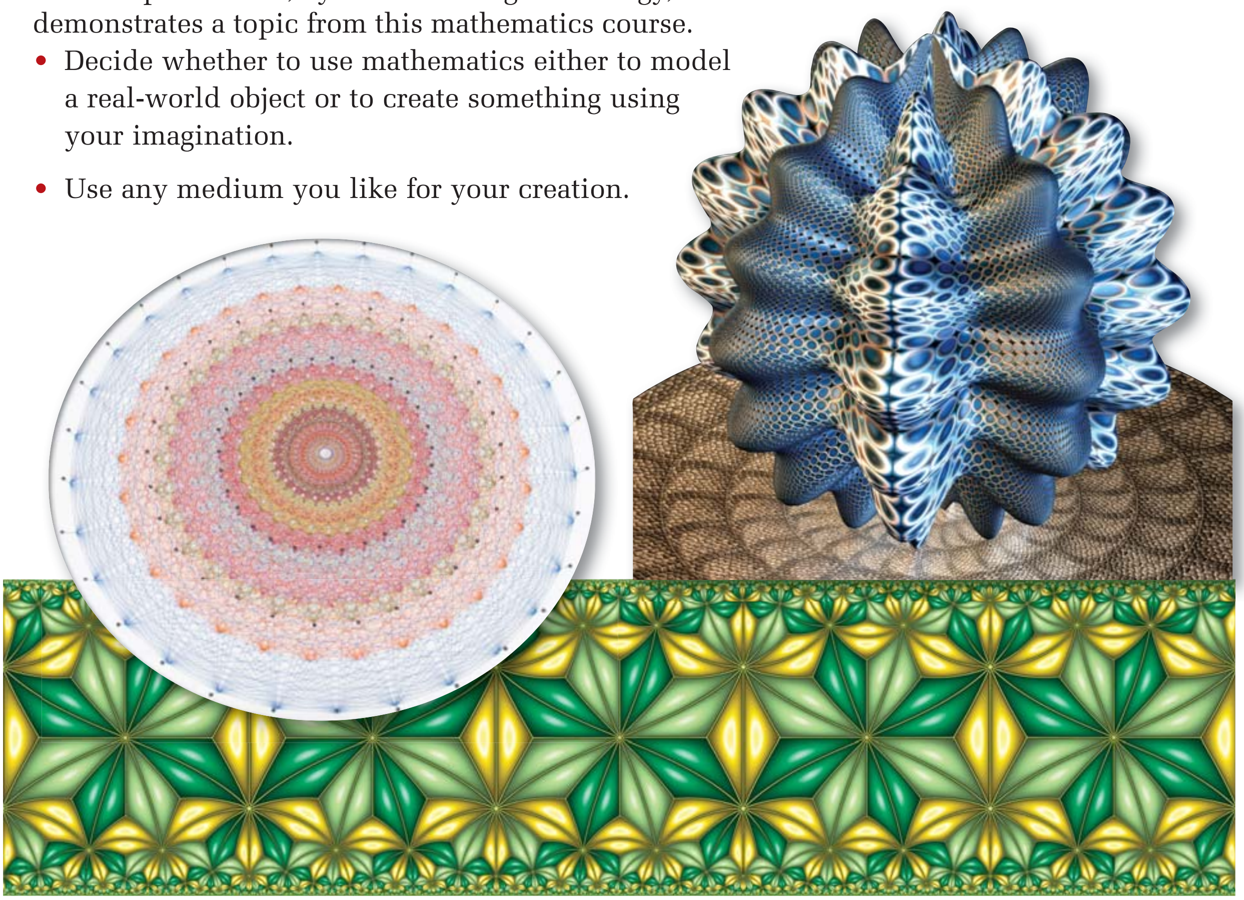
b) Identify the strengths and the weaknesses of each method.

C4 Add to your mathematics career file for this chapter. Identify an occupation or career requiring the use of the binomial theorem. Create at least two problems that could apply to someone working in the chosen occupation or career. Explain how your problems relate to the occupation or career.

Project Corner — Art Presentation

Create a piece of art, by hand or using technology, that demonstrates a topic from this mathematics course.

- Decide whether to use mathematics either to model a real-world object or to create something using your imagination.
- Use any medium you like for your creation.



Chapter 11 Review

11.1 Permutations, pages 516–527

1. A young couple plans to have three children.

a) Draw a tree diagram to show the possible genders for three children.

b) Use your tree diagram to determine the number of outcomes that give them one boy and two girls.

2. A football stadium has nine gates: four on the north side and five on the south side.

a) In how many ways can you enter and leave the stadium?

b) In how many ways can you enter through a north gate and leave by any other gate?

3. How many different arrangements can you make using all of the letters of each word?

a) bite

b) bitten

c) mammal

d) mathematical (leave this answer in factorial form)

4. Five people, Anna, Bob, Cleo, Dina, and Eric, are seated in a row. In how many ways can they be seated if

a) Anna and Cleo must sit together?

b) Anna and Cleo must sit together and so must Dina and Eric?

c) Anna and Cleo must not sit together?

5. In how many ways can the letters of *olympic* be arranged if

a) there are no restrictions?

b) consonants and vowels (o, i, and y) alternate?

c) all vowels are in the middle of each arrangement?

6. Passwords on a certain Web site can have from four to eight characters. A character can be any digit or letter. Any password can have at most one digit on this Web site. Repetitions are allowed.

a) How many four-character passwords are possible?

b) How many eight-character passwords are possible?

c) If a hacker can check one combination every 10 s, how much longer does it take to check all of the eight-character passwords than to check all of the four-character passwords?

7. Simplify each expression.

a) $\dfrac{n! + (n-1)!}{n! - (n-1)!}$

b) $\dfrac{(x+1)! + (x-1)!}{x!}$

11.2 Combinations, pages 528–536

8. Imagine that you have ten small, coloured-light bulbs, of which three are burned out.

a) In how many ways can you randomly select four of the light bulbs?

b) In how many ways can you select two good light bulbs and two burned out light bulbs?

9. Calculate the value of each expression.

a) ${}_{10}C_3$

b) ${}_{10}P_4$

c) ${}_5C_3 \times {}_5P_2$

d) $\left(\dfrac{15!}{4!11!}\right)({}_6P_3)$

10. a) How many different sums of money can you form using one penny, one nickel, one dime, and one quarter?

b) List all possible sums to confirm your answer from part a).

11. Solve for n. Show that each answer is correct.

a) ${}_nC_2 = 28$

b) ${}_nC_3 = 4({}_nP_2)$

12. Ten students are instructed to break into a group of two, a group of three, and a group of five. In how many ways can this be done?

13. a) Create two problems where the answer to each is represented by $\left(\frac{5!}{2!3!}\right)$. One problem must involve a permutation and the other a combination. Explain which is which.

b) Five colours of paint are on sale. Is the number of ways of choosing two colours from the five options the same as the number of ways of choosing three colours from the five? Explain your answer.

11.3 The Binomial Theorem, pages 537–545

14. For each row from Pascal's triangle, write the next row.

a) 1 2 1

b) 1 8 28 56 70 56 28 8 1

15. Explain how to determine the coefficients of the terms in the expansion of $(x + y)^n$, $n \in N$, using multiplication, Pascal's triangle, or combinations. Use examples to support your explanations.

16. Expand using the binomial theorem. Simplify.

a) $(a + b)^5$ **b)** $(x - 3)^3$

c) $\left(2x^2 - \frac{1}{x^2}\right)^4$

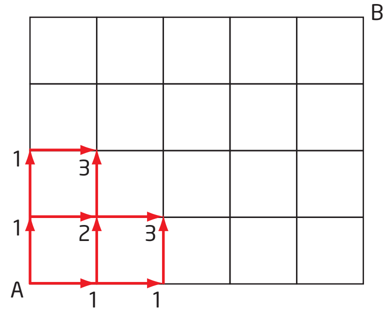
17. Determine the indicated term in each binomial expansion. Simplify each answer.

a) third term of $(a + b)^9$

b) sixth term of $(x - 2y)^6$

c) middle term of $\left(\frac{1}{x} - 2x^2\right)^6$

18. You can determine the number of possible routes from A to each intersection in the diagram. Assume you can move only up or to the right.



a) Draw a diagram like the one shown. Mark the number of possible routes or pathways from A to each intersection in your diagram. A few have been done for you. Use counting and patterns.

b) Show how you can superimpose Pascal's triangle to get the number of pathways from A to any point on the grid.

c) How many pathways are possible to go from A to B?

d) Use a different method to determine the number of possible pathways going from A to B.

19. Ten green and ten yellow counters are placed alternately in a row.

On each move, a counter can only jump one of its opposite colour. The task is to arrange all of the counters in two groups so that all of the green counters are at one end and all of the yellow counters are at the other end.

a) How many moves are necessary if you have ten green and ten yellow counters?

b) Establish a pattern for 2 to 12 counters of each colour. Use 2 different colours of counters.

c) How many moves are necessary for 25 of each colour?

Chapter 11 Practice Test

Multiple Choice

For #1 to #6, choose the best answer.

1. How many three-digit numbers with no repeating digits can be formed using the digits 0, 1, 2, 8, and 9?

A 100 **B** 60 **C** 48 **D** 125

2. In how many ways can the letters of the word SWEEPERS be arranged in a row?

A 40 320 **B** 20 160

C 6720 **D** 3360

3. How many five-member committees containing two Conservatives, two New Democrats, and one Liberal can be formed from seven Conservatives, six New Democrats, and five Liberals?

A 6300 **B** 3150 **C** 1575 **D** 8568

4. How many terms are in the expansion of $(2x - 5y^2)^{11}$?

A 13 **B** 12 **C** 11 **D** 10

5. What is a simplified expression for the third term in the expansion of $(2x^2 + 3y)^7$?

A $6048x^{10}y^2$ **B** $9072x^8y^3$

C $12\ 096x^{10}y^2$ **D** $15\ 120x^8y^3$

6. The numbers 1 6 15 20 15 6 1 represent the seventh row in Pascal's triangle. What is the sixth number in the next row?

A 1 **B** 7 **C** 21 **D** 35

Short Answer

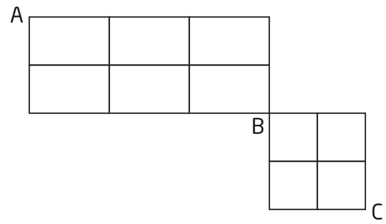
7. For six multiple choice questions, two answers are A, two answers are B, one answer is C, and one answer is D.

a) How many answer keys are possible?

b) List the possible answer keys if you know that the answers to questions 3 and 5 are C and D, respectively.

8. Carla claims that when you solve $_nP_2 = 72$, there are two possible answers, and that one of the answers is −8. Do you agree with her? Explain.

9. Consider the diagram.



a) How many pathways from A to B are possible, moving only down or to the right?

b) How many pathways from A to C are possible, moving only down or to the right? Explain using permutations and the fundamental counting principle.

10. How many numbers of at most three digits can be created from the digits 0, 1, 2, 3, and 4?

11. Explain, using examples, the difference between a permutation and a combination.

12. Determine the simplified term that contains x^9 in the expansion of $\left(x^2 + \frac{2}{x}\right)^9$.

Extended Response

13. a) How many 4-digit even numbers greater than 5000 can you form using the digits 0, 1, 2, 3, 5, 6, 8, and 9 without repetitions?

b) How many of these numbers end in 0?

14. Solve for n.

a) $_nP_3 = 120$

b) $3(_nC_2) = 12(_nC_1)$

15. Expand $\left(y - \frac{2}{y^2}\right)^5$ using the binomial theorem. Simplify your answer.

16. How many ways can all the letters of *aloha* be arranged if

a) the *a*'s must be together?

b) the *a*'s cannot be together?

c) each arrangement must begin with a vowel and the consonants cannot be together?